RELATION BETWEEN PASCAL'S TRIANGLE AND VOLUME OF HYPERCUBES

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Abstract. In this short report famous binomial identity

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

is generalized and relation between Pascal's triangle and volume of m-dimension n-length Hypercube is shown, where $(m,\ n)$ are positive integers.

1. Introduction and main results

In this section let review and generalize well known fact about connection between row sums of Pascal triangle and $2-$dimension Hypercube, recall property

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n \tag{1.1}$$

Now, let multiply each k-th term of of n-th row of Pascal's triangle [1] by 2^k

$$\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & & 2 & & & \\
 & & 1 & & 4 & & 4 & & \\
 & 1 & & 6 & & 12 & & 8 & \\
1 & & 8 & & 24 & & 32 & & 16
\end{array}$$

Figure 5. Triangle built by $\binom{n}{k} \cdot 2^k,\ 0 \leq k \leq n \leq 4$.

We can notice that

$$\sum_{k=0}^{n} \binom{n}{k} \cdot 2^k = 3^n, \quad 0 \leq k \leq n, \quad (n,\ k) \in \mathbb{N} \tag{1.2}$$

Volume of n-dimension hypercube with length m could be calculated as

$$m^n = \sum_{k=0}^{n} \sum_{j=0}^{k} \binom{n}{k} \binom{k}{j} (-1)^{k-j} m^j \tag{1.3}$$

where m and n - positive integers, see [5].

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Proof. Recall induction over m, in (1.1) is shown a well-known example for $m = 2$.

$$2^n = \sum_{k=0}^{n} \binom{n}{k} (2-1)^k \tag{1.4}$$

Review (1.4) and suppose that

$$\underbrace{(2+1)}_{m=3}{}^n = \sum_{k=0}^{n} \binom{n}{k} \cdot (\underbrace{(2-1)+1}_{m-1})^k \tag{1.5}$$

And, obviously, this statement holds by means of Newton's Binomial Theorem [2], [3] given $m = 3$, more detailed, recall expansion for $(x+1)^n$ to show it.

$$(x+1)^n = \sum_{k=0}^{n} \binom{n}{k} x^k \tag{1.6}$$

Substituting $x = 2$ to (1.6) we have reached (1.5).
Next, let show example for each $m \in \mathbb{N}$. Recall Binomial theorem to show this

$$m^n = \sum_{k=0}^{n} \binom{n}{k} \cdot (m-1)^k \tag{1.7}$$

Hereby, for $m+1$ we receive Binomial theorem again

$$(m+1)^n = \sum_{k=0}^{n} \binom{n}{k} \cdot m^k \tag{1.8}$$

Review result from (1.7) and substituting Binomial expansion $\sum_{j=0}^{k} \binom{k}{j} (-1)^{n-k} m^j$ instead $(m-1)^k$ we receive desired result

$$\begin{aligned} m^n &= \sum_{k=0}^{n} \binom{n}{k} \cdot \underbrace{(m-1)^k}_{\sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} m^j} = \sum_{k=0}^{n} \binom{n}{k} \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} m^j \\ &= \sum_{k=0}^{n} \sum_{j=0}^{k} \binom{n}{k} \binom{k}{j} (-1)^{k-j} m^j \end{aligned} \tag{1.9}$$

This completes the proof. □

Number of elements $k-$face elements $\mathscr{E}_k(\mathbf{Y}_n^p)$ of Generalized Hypercube $\mathbf{Y}_n^p$ equals to

$$\mathscr{E}_k(\mathbf{Y}_n^p) = \sum_{j=0}^{k} \binom{n}{k} \binom{k}{j} (-1)^{k-j} (p-1)^j \tag{1.10}$$

To verify the expression (1.9) use Mathematica code:

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In[19]:= Sum[Sum[Binomial[n,k]*Binomial[k,j]*(−1)^(k−j)*m^j,
         {j, 0, k}],{k, 0, n}]
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It could be found at .txt file in the last line here.

References

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