ON THE NUMERICAL EXPANSION OF MONOMIALS

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Abstract. In this short report particular pattern, that is sequence A287326 in OEIS, [4], [3], [2], which shows us necessary items to expand $x^3,\ x \in \mathbb{N}$ will be generalized and obtained results will be applied to show expansion of power function $f(x) = x^n,\ (x, n) \in \mathbb{N}$. In Section 3 received results are used to obtain finite differences of power function $f(x)$.

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1. Introduction and Main Results

In recent work ([3], eq. 1.15) for each positive integer x was shown the identity

$$x^3 = x + (x-0)\cdot 3!\cdot 0 + (x-1)\cdot 3!\cdot 1 + (x-2)\cdot 3!\cdot 2 + \cdots \tag{1.1}$$

$$\cdots + (x-(x-1))\cdot 3!\cdot (x-1)$$

Particularizing expression (1.3) and applying compact sigma notation, one could have

$$x^3 = \sum_{m=0}^{x-1} 3!\cdot mx - 3!\cdot m^2 + 1, \qquad x \in \mathbb{N} \tag{1.2}$$

Lets build a triangle using $3!\cdot kn - 3!\cdot k^2 + 1$ over k and n, where n - denotes the row, k - corresponding item of the row

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$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & & 1 & & & \\ & & 1 & & 7 & & 1 & & \\ & 1 & & 13 & & 13 & & 1 & \\ 1 & & 19 & & 25 & & 19 & & 1 \end{array} \tag{1.3}$$

Figure 1. Triangle generated by $3! \cdot kx - 3! \cdot k^2 + 1, \ 0 \leq k \leq n = 4$, sequence A287326 in OEIS, [4], [3], [2].

The main property of above triangle is

Property 1.4. *Summation of each n-th row of Triangle (1.2) from $k = 0$ to $n - 1$ returns us n^3.*

Reader could notice that Triangle (1.2) has similar distribution to Pascal's triangle [1], [5], hereby the follow questions is stated:

Question 1.5. *Has the Triangle (1.2) any connection with Pascal's Triangle, and is it exist similar patterns in order to receive expansion of x^j $j > 3$?*

To answer to the question (1.4), let rewrite and review our Triangle (1.2) again

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & & 1 & & & \\ & & 1 & & 7 & & 1 & & \\ & 1 & & 13 & & 13 & & 1 & \\ 1 & & 19 & & 25 & & 19 & & 1 \end{array} \tag{1.6}$$

Figure 2. Triangle (1.2).

Let take away from each item k, such that $0 < k < n$ of Triangle (1.5) the value of n^2, then we have

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & & 1 & & & \\ & & 1 & & 3 & & 1 & & \\ & 1 & & 4 & & 4 & & 1 & \\ 1 & & 3 & & 9 & & 3 & & 1 \end{array} \tag{1.7}$$

Figure 3. Triangle generated by $\begin{cases} 3! \cdot kn - 3! \cdot k^2 + 1 - n^2, \ 0 < k < n \\ 1, \ k \in \{0, \ n\} \end{cases}$

We can observe that summation of n-th row of Triangle (1.6) over k from 0 to $n-1$ returns us the n^2. It's very easy to see that removing n^1 from each item k, such that $0 < k < n$ of Triangle (1.6) and summing up received n-th rows over k from 0 to $n - 1$ will result n^1, let show it

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & & 1 & & & \\ & & 1 & & 1 & & 1 & & \\ & 1 & & 1 & & 1 & & 1 & \\ 1 & & -1 & & 5 & & -1 & & 1 \end{array} \tag{1.8}$$

Figure 4. Triangle generated by $\begin{cases} 3! \cdot kn - 3! \cdot k^2 + 1 - n^2 - n^1, \ 0 < k < n \\ 1, \ k \in \{0, \ n\} \end{cases}$

Review the Triangle (1.7), we can say that above statement holds. Reviewing our Triangles (1.2), (1.6), (1.7), let define generalized item $V_M(n, \ k)$

Definition 1.9.

$$V_M(n, \ k) := \begin{cases} n^0 + n^1 + \cdots + n^M, \ 0 < k < n \\ 1, \ k \in \{0, \ n\} \end{cases} \tag{1.10}$$

Property 1.11. *From definition (1.8) follows the equality between items $V_M(n, \ k)$ in range $k \in \{1, \ n-1\}$*

$$\begin{aligned} \forall k \in \{1, \ n-1\} : \ V_M(n, \ k) &= V_M(n, \ k+1 \leq n-1) \\ &= V_M(n, \ k+2 \leq n-1) \\ &\cdots \\ &= V_M(n, \ k+j \leq n-1), \ j \in \mathbb{N} \end{aligned} \tag{1.12}$$

Note that k-th items of Triangles (1.2), (1.6), (1.7), such that $0 < k < n$ are $V_0(n, \ k)$, $V_1(n, \ k)$, $V_2(n, \ k)$, respectively. Reviewing our triangles Triangles (1.2), (1.6), (1.7), we could observe the identity

$$n^M = \sum_{k=0}^{n-1} V_{M-1}(n, \ k), \qquad M \in \{1, 2, 3\} \tag{1.13}$$

Example 1.14. Review (1.13), let be $n = 4, \ M = 3$, then

$$\begin{aligned} 4^3 &= V_2(4, \ 0) + V_2(4, \ 1) + V_2(4, \ 2) + V_2(4, \ 3) \\ &= 1 + \underbrace{1 + 4 + 4^2}_{V_2(4, \ 1)} + \underbrace{1 + 4 + 4^2}_{V_2(4, \ 2)} + \underbrace{1 + 4 + 4^2}_{V_2(4, \ 3)} \\ &= 1 + 3(4^0 + 4^1 + 4^2) = 1 + 21 + 21 + 21 \end{aligned} \tag{1.15}$$

Let be Theorem

Theorem 1.16. *Each power function $f(x) = x^n$ such that $(x, n) \in \mathbb{N}$ could be expanded next way*

$$x^n = \sum_{k=0}^{x-1} V_{n-1}(x, \ k), \qquad \forall (x, \ n) \in \mathbb{N} \tag{1.17}$$

Proof. Recall Triangle, consisting of items $V_0(x, \ k)$, that is analog of (1.7)

$$
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & & 1 & & & \\
 & & 1 & & 1 & & 1 & & \\
 & 1 & & 1 & & 1 & & 1 & \\
1 & & 1 & & 1 & & 1 & & 1
\end{array}
\tag{1.18}
$$

Figure 5. Triangle generated by $V_0(x,\ k)$, sequence A000012 in OEIS, [6], [2].

Obviously, summation over rows of triangle (1.16) from 0 to $x-1$ gives us the x^1. Let show a second power by means of $V_0(x,\ k)$, we have accordingly

$$
x^2 = 1 + \underbrace{(V_0(x,\ 1) + x)}_{V_1(x,\ 1)} + \underbrace{(V_0(x,\ 2) + x)}_{V_1(x,\ 2)} + \cdots + \underbrace{(V_0(x,\ x-1) + x)}_{V_1(x,\ x-1)} \tag{1.19}
$$

For example, consider the sum of 3-rd row of triangle A000012, then we receive 3^1, then by power property

$$
x^n = \sum_{k=0}^{x-1} x^{n-1}
$$

we have to add twice by 3^1 to receive 3^2, ie

$$
3^2 = 1 + (3+1) + (3+1)
$$

Generalizing above result, we have identity

$$
\begin{aligned}
\sum_{k=0}^{x-1} V_{n-1}(x,\ k) &= \\
&= 1 + \underbrace{(x^0 + x^1 + \cdots + x^{n-1}) + \cdots + (x^0 + x^1 + \cdots + x^{n-1})}_{x-1 \text{ times}} \\
&= 1 + (x-1)(x^0 + x^1 + x^2 + \cdots + x^{n-1}) = 1 + (x-1)V_{n-1}(x,\ k) \qquad (1.20)\\
&= x + (x-1)x + (x-1)x^2 + (x-1)x^3 + \cdots + (x-1)x^{n-1} \\
&= 1 + x^n - x^0 = x^n,\ (x,n) \in \mathbb{N}
\end{aligned}
$$

This completes the proof. □

Also, (1.18) could be rewritten as

$$
\begin{aligned}
x^n &= 1 + \underbrace{(x-1) + (x-1)x + \cdots + (x-1)x^{n-1}}_{n \text{ times}} \qquad (1.21)\\
&= 1 + \sum_{k=0}^{n-1} (x-1)x^k,\ \forall (x,\ n) \in \mathbb{N} \qquad (1.22)
\end{aligned}
$$

Define the power function $f(x) = x^n$, such that $(x,\ n) \in \mathbb{N}$ and exponential function $g(x,\ k) = x^k,\ x \in \mathbb{N}$, then expression (1.20) shows us the relation between exponential and power functions with natural base and exponent

$$
f(x) = 1 + \sum_{k=0}^{n-1} g(x,\ k+1) - g(x,\ k) \tag{1.23}
$$

Reader could also notice the connection between Maclaurin expansion $\frac{1}{1-x} = x^0 + x^1 + x^2 + x^3 + \cdots,\ -1 < x < 1$ and (1.18), that is

$$\frac{x^n}{x-1} - \underbrace{\frac{1}{x-1}}_{-\frac{1}{1-x}} = x^0 + x^1 + x^2 + \cdots + x^{n-1},\ \forall (x,\ n) \in \mathbb{N} \tag{1.24}$$

Next, let review and apply our results on Binomial Related expansion of monomial $f(x) = x^n,\ (x,\ n) \in \mathbb{N}$, that is summation of finite difference of power with increment $h = 1$

$$\begin{aligned} x^n &= \sum_{k=0}^{x-1} \Delta_h[x^n] \\ &= \sum_{k=0}^{x-1} \underbrace{nk^{n-1} + \binom{n}{2}k^{n-2} + \cdots + \binom{n}{n-1}k + 1}_{\Delta_h[x^n]=(x+h)^n - x^n} \\ &= \sum_{j=0}^{x-1}\sum_{k=1}^{n} \binom{n}{k} j^{n-k},\ x \in \mathbb{N} \end{aligned} \tag{1.25}$$

Then, it follows for each $(x, m, n) \in \mathbb{N}$

$$\begin{aligned} x^{n+1} &= \sum_{j=0}^{x-1}\sum_{k=1}^{n+1} \binom{n}{k} j^{n+1-k} = \sum_{j=0}^{x-1}\sum_{k=1}^{n} \binom{n}{k} j^{n-k} + x^n \\ x^{n+m} &= \sum_{j=0}^{x-1}\sum_{k=1}^{n} \binom{n}{k} j^{n-k} + x^n + x^{n+1} + \cdots + x^{n+m-1},\ x \in \mathbb{N} \end{aligned} \tag{1.26}$$

To show one more property of $V_M(n,\ k)$, let build Triangle given $V_2(n,\ k)$

$$\begin{array}{ccccccccccccccc} & & & & & & & 1 & & & & & & & \\ & & & & & & 1 & & 1 & & & & & & \\ & & & & & 1 & & 3 & & 1 & & & & & \\ & & & & 1 & & 4 & & 4 & & 1 & & & & \\ & & & 1 & & 5 & & 5 & & 5 & & 1 & & & \\ & & 1 & & 6 & & 6 & & 6 & & 6 & & 1 & & \\ & 1 & & 7 & & 7 & & 7 & & 7 & & 7 & & 1 & \\ 1 & & 8 & & 8 & & 8 & & 8 & & 8 & & 8 & & 1 \end{array}$$

Figure 6. Triangle generated by $V_2(n,\ k)$ over n from 0 to 9.

Reviewing above triangle, we could observe that summation of intermediate column gives us well known identity,

$$x^2 = \sum_{k=0}^{|x|-1} 2k + 1,\ x \in \mathbb{Z} \tag{1.27}$$

Generalizing above expression, we receive

$$x^n = \sum_{k=0}^{n-1} V_{n-1}(2k,\ k),\ \forall (x,\ n) \in \mathbb{N} \tag{1.28}$$

More detailed,

$$V_n(2k,\ k) = \begin{cases} (2k)^0 + (2k)^1 + \cdots + (2k)^n,\ 0 < k < n \\ 1,\ k \in \{0,\ n\} \end{cases} \tag{1.29}$$

Then

$$\sum_{k=0}^{n-1} V_{n-1}(2k,\ k) =$$

$$= 1 + (1 + 2 + \cdots + 2^{n-1}) + (1 + 2 \cdot 2^1 + \cdots + (2 \cdot 2)^{n-1}) \tag{1.30}$$

$$+ (1 + 2 \cdot 3 \cdots + (2 \cdot 3)^{n-1}) + \cdots \tag{1.31}$$

$$+ (1 + 2 \cdot (n-1) \cdots + (2 \cdot (n-1))^{n-1}) = x^n \tag{1.32}$$

Note that upper expression (1.26) is partial case of (1.24), when $n + m = 2$. Recall the Binomial $(x + y)^n$, by means of (1.19) we have expansion

$$(x + y)^n = 1 + (x + y - 1)V_{n-1}(x + y,\ k) \tag{1.33}$$

Hereby, let be lemma

Lemma 1.34. *Relation between binomial expansion and* $V_{n-1}(x,\ k)$

$$(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k = 1 + (x + y - 1)V_{n-1}(x + y,\ k) \tag{1.35}$$

$$= 1 + (x + y - 1)(x + y)^0 + (x + y - 1)(x + y)^1 + \cdots \tag{1.36}$$

$$+ (x + y - 1)(x + y)^{n-1} \tag{1.37}$$

$$= 1 + x((x + y)^0 + (x + y)^1 + (x + y)^2 + \cdots + (x + y)^{n-1}) \tag{1.38}$$

$$+ y((x + y)^0 + (x + y)^1 + (x + y)^2 + \cdots + (x + y)^{n-1}) \tag{1.39}$$

$$- ((x + y)^0 + (x + y)^1 + (x + y)^2 + \cdots + (x + y)^{n-1}) \tag{1.40}$$

Multinomial case could be built as well as Binomial, hereby

$$(x_1 + x_2 + \cdots + x_k) = \tag{1.41}$$

$$= 1 + (x_1 + x_2 + \cdots + x_k - 1)(x_1 + x_2 + \cdots + x_k)^0 \tag{1.42}$$

$$+ (x_1 + x_2 + \cdots + x_k - 1)(x_1 + x_2 + \cdots + x_k)^1 \tag{1.43}$$

$$+ (x_1 + x_2 + \cdots + x_k - 1)(x_1 + x_2 + \cdots + x_k)^2 \tag{1.44}$$

$$\vdots \tag{1.45}$$

$$+ (x_1 + x_2 + \cdots + x_k - 1)(x_1 + x_2 + \cdots + x_k)^{n-1} \tag{1.46}$$

2. Finite Differences

In this section let apply received in previous section results to show finite differences of power function $f(x) = x^n$, such that $(x,\ n) \in \mathbb{N}$. From (1.19) we know identity

$$f(x) = 1 + (x - 1)(x^0 + x^1 + x^2 + \cdots + x^{n-1}) = 1 + (x - 1)V_{n-1}(x,\ k) = x^n \tag{2.1}$$

Then, its finite difference $\Delta f(x)$ suppose to be

$$
\begin{aligned}
(2.2)\quad \Delta f(x) &= f(x+1) - f(x) = \\
(2.3)\quad &= [1 + \underbrace{x((x+1)^0 + (x+1)^1 + (x+1)^2 + \cdots + (x+1)^{n-1})}_{xV_{n-1}(x+1,\ k)}] \\
(2.4)\quad &- [1 + \underbrace{(x-1)(x^0 + x^1 + x^2 + \cdots + x^{n-1})}_{(x-1)V_{n-1}(x,\ k) = x^n - 1}] \\
(2.5)\quad &= xV_{n-1}(x+1,\ k) - (x-1)V_{n-1}(x,\ k) \\
(2.6)\quad &= xV_{n-1}(x+1,\ k) - x^n - 1 \\
(2.7)\quad &= xV_{n-1}(x+1,\ k) - xV_{n-1}(x,\ k) + V_{n-1}(x,\ k) \\
(2.8)\quad &= x[V_{n-1}(x+1,\ k) - V_{n-1}(x,\ k)] + V_{n-1}(x,\ k),\ k \notin \{0,\ n\}
\end{aligned}
$$

For example, let be $x = 3,\ n = 3$, then we have

$$
\begin{aligned}
(2.9)\quad \Delta f(3) &= f(4) - f(3) \\
(2.10)\quad &= [1 + 3((3+1)^0 + (3+1)^1 + (3+1)^2)] \\
(2.11)\quad &- [1 + (3-1)(3^0 + 3^1 + 3^2)] \\
(2.12)\quad &= [3((3+1)^0 + (3+1)^1 + (3+1)^2)] \\
(2.13)\quad &- [(3-1)(3^0 + 3^1 + 3^2)] \\
(2.14)\quad &= 63 - 26 = 37
\end{aligned}
$$

Let show high order finite difference of power $f(x) = x^n$ by means of $V_{n-1}(x,\ k)$, that is

$$
(2.15)\quad \Delta^m f(x) = \sum_{k=0}^{m-1} (x-k)[V_{n-1}(x+m-k,\ t \neq 0) - V_{n-1}(x+m-k+1,\ t \neq 0)]
$$

Derivative of $f(x) = x^n$ could be written regarding to

$$
(2.16)\quad \frac{df(x)}{dx} = \lim_{h \to 0} \left[\frac{xV_{n-1}(x+1,\ k) - (x-1)V_{n-1}(x,\ k)}{h} \right]
$$

3. Conclusion

In this paper particular pattern, that is sequence A287326 in OEIS, [4], [3], [2]., which shows us necessary items to expand $x^3,\ x \in \mathbb{N}$ was generalized and obtained results were applied to show expansion of $f(x) = x^n,\ (x, n) \in \mathbb{N}$. Additionally, relation between exponential and power functions with natural base and exponent was shown by means of expression (1.20).

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