ON THE LINK BETWEEN BINOMIAL THEOREM AND DISCRETE CONVOLUTION OF POWER FUNCTION

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Abstract. In this manuscript we establish a relation between Binomial theorem and discrete convolution of piecewise defined power function. We show that Binomial expansion of s-powered sum $(x+y)^s, \ s \geq 1, \ x+y \geq 1$ is equivalent to the sum of product of discrete convolutions of power function and certain real coefficients within the finite interval of positive integers. In addition, we generalise relation between Binomial theorem and discrete convolution of piecewise defined power function to Multinomial case.

Contents

1. Introduction

In this paper we reveal a relation between famous Binomial theorem [AS72] and discrete convolution [BDM11] of piecewise defined power function $\langle x \rangle^n$, where angle brackets denotes the Macaulay convention, see (1.1) for $\langle x \rangle^n$. The content of the manuscript reaches the main aim of the work through the following milestones. Firstly, we perform a detailed discussion on $(2m+1)$-degree integer-valued polynomials $\mathbf{P}_b^m(n)$. We show all the implicit forms of

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the polynomials $\mathbf{P}^m_b(n)$ and discuss their main properties. Finally, for the first milestone, we arrive to the identity between odd-powered Binomial (and Multinomial) expansions and partial case of $\mathbf{P}^m_b(n)$. As next step, we establish a relation between the polynomials $\mathbf{P}^m_b(n)$ and discrete convolution of the power function $\langle x \rangle^n$. This relation is consequence of the following claims:

- $\mathbf{P}^m_b(n)$ is in relation with the convolutional power sum $\mathbf{C}^r_n(b)$, see (1.1) for $\mathbf{C}^r_n(b)$.
- Discrete convolution of power function $\langle x \rangle^n$ is partial case of the power sum $\mathbf{C}^r_n(b)$.
- Polynomials $\mathbf{P}^m_b(n)$ are in relation with the discrete convolution of power function $\langle x \rangle^n$.

Then, in the subsection (3.1) we particularise obtained results to show the relation between Binomial (and Multinomial) theorem and the discrete convolution of piecewise defined power function.

1.1. **Definitions, Notations and Conventions.** We now set the following notation, which remains fixed for the remainder of this paper:

- $\mathbf{A}_{m,r}$ is a real coefficient defined recursively

$$\mathbf{A}_{m,r} := \begin{cases} (2r+1)\binom{2r}{r}, & \text{if } r = m \\ (2r+1)\binom{2r}{r}\sum_{d=2r+1}^{m} \mathbf{A}_{m,d}\binom{d}{2r+1}\frac{(-1)^{d-1}}{d-r}B_{2d-2r}, & \text{if } 0 \leq r < m \\ 0, & \text{if } r < 0 \text{ or } r > m \end{cases}$$

  where B_t are Bernoulli numbers [Wei]. We assume that $B_1 = \frac{1}{2}$.

- $\mathbf{L}_m(n,k)$ is polynomial of degree $2m$ in n, k

$$\mathbf{L}_m(n,k) := \sum_{r=0}^{m} \mathbf{A}_{m,r} k^r (n-k)^r$$

- $\mathbf{P}^m_b(n)$ is polynomial of degree $2m+1$ in b, n

$$\mathbf{P}^m_b(n) := \sum_{k=0}^{b-1} \mathbf{L}_m(n,k)$$

- $\mathbf{C}^r_n(b)$ is a convolutional power sum

$$\mathbf{C}^r_n(b) := \sum_{k=0}^{b-1} k^r (n-k)^r$$

- $\mathbf{H}_{m,t}(b)$ is a real coefficient defined as

$$\mathbf{H}_{m,t}(b) := \sum_{j=t}^{m} \binom{j}{t} \mathbf{A}_{m,j} \frac{(-1)^j}{2j-t+1} \binom{2j-t+1}{b} B_{2j-t+1-b}$$

- $\mathbf{X}_{m,t}(j)$ is polynomial of degree $2m-t$ in b

$$\mathbf{X}_{m,t}(j) := (-1)^m \sum_{k=1}^{2m-t+1} \mathbf{H}_{m,t}(k) \cdot j^k$$

- $\mathbf{S}_p(n)$ is a common power sum
$$\mathbf{S}_p(n) := \sum_{k=0}^{n-1} k^p$$
- We believe to [GKP94] that exponential function 0^x should be defined for all x as
$$0^x = 1$$
- $[P(k)]$ is the Iverson's convention [Ive62], where $P(k)$ is logical sentence depending on k
$$[P(k)] = \begin{cases} 1, & P(k) \text{ is true} \\ 0, & \text{otherwise} \end{cases}$$
- $\langle x-a\rangle^n$ is powered Macaulay bracket, [Mac19]
$$\langle x-a\rangle^n := \begin{cases} (x-a)^n, & x \geq a \\ 0, & \text{otherwise} \end{cases} \quad a \in \mathbb{Z}$$
- $\{x-a\}^n$ is powered Macaulay bracket
$$\{x-a\}^n := \begin{cases} (x-a)^n, & x > a \\ 0, & \text{otherwise} \end{cases} \quad a \in \mathbb{Z}$$
During the manuscript, the variable a is reserved to be only the condition of Macaulay functions. If the power function $\langle x-a\rangle^n$ or $\{x-a\}^n$ is written without parameter a e.g $\langle x+y-t\rangle^n$ means that it is assumed $a=0$.

2. Discussion on the Polynomials $\mathbf{P}_b^m(n)$

We'd like to begin our discussion from defined above polynomial $\mathbf{P}_b^m(n)$. Polynomial $\mathbf{P}_b^m(n)$ is $2m+1$ degree polynomial in b, n. Polynomial $\mathbf{P}_b^m(n)$ is defined as finite sum of $2m$ degree polynomial $\mathbf{L}_m(n,k)$ over k from zero to $b-1$. The $2m$-degree polynomials $\mathbf{L}_m(n,k)$ are to be in extended form as
$$\mathbf{L}_m(n,k) = \mathbf{A}_{m,m}k^m(n-k)^m + \mathbf{A}_{m,m-1}k^{m-1}(n-k)^{m-1} + \cdots + \mathbf{A}_{m,0},$$
where $\mathbf{A}_{m,r}$ are real coefficients. The coefficients $\mathbf{A}_{m,r}$ are nonzero only for integer r within the interval $r \in \{m\} \cup [0, \lfloor \frac{m-1}{2} \rfloor$. For instance, let show an example of $\mathbf{A}_{m,r}$ coefficients

m/r	0	1	2	3	4	5	6	7
0	1							
1	1	6						
2	1	0	30					
3	1	-14	0	140				
4	1	-120	0	0	630			
5	1	-1386	660	0	0	2772		
6	1	-21840	18018	0	0	0	12012	
7	1	-450054	491400	-60060	0	0	0	51480

Table 1. Coefficients $\mathbf{A}_{m,r}$.

For $m \geq 11$ the $\mathbf{A}_{m,r}$ could be a fractional number. Thus, the polynomial $\mathbf{L}_m(n,k)$ could be written in the form

$$\mathbf{L}_m(n,k) = \mathbf{A}_{m,m} k^m (n-k)^m + \sum_{r=0}^{\lfloor \frac{m-1}{2} \rfloor} \mathbf{A}_{\lfloor \frac{m-1}{2} \rfloor, r} k^r (n-k)^r$$

For example, a few of polynomials $\mathbf{L}_m(n,k)$ are

$$\begin{aligned}
\mathbf{L}_0(n,k) &= 1, \\
\mathbf{L}_1(n,k) &= 6k(n-k) + 1 = -6k^2 + 6kn + 1, \\
\mathbf{L}_2(n,k) &= 30k^2(n-k)^2 + 1 = 30k^4 - 60k^3 n + 30k^2 n^2 + 1, \\
\mathbf{L}_3(n,k) &= 140k^3(n-k)^3 - 14k(n-k) + 1 \\
&= -140k^6 + 420k^5 n - 420k^4 n^2 + 140k^3 n^3 + 14k^2 - 14kn + 1
\end{aligned}$$

Here we briefly discussed the polynomials $\mathbf{L}_m(n,k)$, which is required step to be essentially familiarized with the $\mathbf{P}_b^m(n)$. For now let's back to main discussion concerning $\mathbf{P}_b^m(n)$. The polynomials $\mathbf{P}_b^m(n)$ implicitly involve the convolutional power sum $\mathbf{C}_n^r(b)$, the polynomial $\mathbf{X}_{m,t}(b)$ the real coefficient $\mathbf{H}_{m,t}(b)$ and common power sum $\mathbf{S}_p(n)$. In extended form, the polynomial $\mathbf{P}_b^m(n)$ is following

(2.1)

$$\begin{aligned}
\mathbf{P}_b^m(n) &= \sum_{k=0}^{b-1} \mathbf{L}_m(n,k) = \sum_{k=0}^{b-1} \sum_{r=0}^{m} \mathbf{A}_{m,r} k^r (n-k)^r = \sum_{r=0}^{m} \mathbf{A}_{m,r} \sum_{k=0}^{b-1} k^r (n-k)^r = \sum_{r=0}^{m} \mathbf{A}_{m,r} \mathbf{C}_b^r(n) \\
&= \sum_{r=0}^{m} \mathbf{A}_{m,r} \sum_{k=0}^{b-1} k^r \sum_{j=0}^{r} (-1)^j \binom{r}{j} n^{r-j} k^j = \sum_{r=0}^{m} \sum_{j=0}^{r} (-1)^j n^{r-j} \binom{r}{j} \mathbf{A}_{m,r} \sum_{k=0}^{b-1} k^{r+j} \\
&= \sum_{r=0}^{m} \sum_{j=0}^{r} n^{r-j} \binom{r}{j} \mathbf{A}_{m,r} \frac{(-1)^j}{r+j+1} \sum_{s=0}^{r+j} \binom{r+j+1}{s} B_s (b-1)^{r+j-s+1}
\end{aligned}$$

However, by the symmetry of $\mathbf{L}_m(n,k)$,

$$\mathbf{L}_m(n,k) = \mathbf{L}_m(n-k,k)$$

the $\mathbf{P}_b^m(n)$ could be written in the form

$$\begin{aligned}
\mathbf{P}_b^m(n) &= \sum_{k=1}^{b} \mathbf{L}_m(n,k) = \sum_{k=1}^{b} \sum_{r=0}^{m} \mathbf{A}_{m,r} k^r (n-k)^r = \sum_{k=1}^{b} \sum_{r=0}^{m} \mathbf{A}_{m,r} k^r \sum_{t=0}^{r} (-1)^{r-t} n^t \binom{r}{t} k^{r-t} \\
&= \sum_{k=1}^{b} \sum_{r=0}^{m} \mathbf{A}_{m,r} k^r \sum_{t=0}^{r} (-1)^{r-t} n^t \binom{r}{t} k^{r-t} = \sum_{t=0}^{m} n^t \underbrace{\sum_{k=1}^{b} \sum_{r=t}^{m} (-1)^{r-t} \binom{r}{t} \mathbf{A}_{m,r} k^{2r-t}}_{(-1)^{m-t} \mathbf{X}_{m,t}(b)} \\
&= \sum_{t=0}^{m} n^t \sum_{r=t}^{m} (-1)^{r-t} \binom{r}{t} \mathbf{A}_{m,r} \mathbf{S}_{2r-t+1}(b)
\end{aligned}$$

From this formula it may be not immediately clear why $\mathbf{X}_{m,t}(b)$ represent polynomials in b. However, this can be seen if we change the summation order again and use Faulhaber's formula to obtain:

$$\mathbf{X}_{m,t}(b) = (-1)^m \sum_{r=t}^{m} \binom{r}{t} \mathbf{A}_{m,r} \frac{(-1)^r}{2r-t+1} \sum_{\ell=0}^{2r-t} \binom{2r-t+1}{\ell} B_\ell b^{2r-t+1-\ell}$$

Introducing $k = 2r - t + 1 - \ell$, we further get the formula

$$\mathbf{X}_{m,t}(b) = (-1)^m \sum_{k=1}^{2m-t+1} b^k \underbrace{\sum_{r=t}^{m} \binom{r}{t} \mathbf{A}_{m,r} \frac{(-1)^r}{2r-t+1} \binom{2r-t+1}{k} B_{2r-t+1-k}}_{\mathbf{H}_{m,t}(k)}$$

It gives us opportunity to review the $\mathbf{P}_b^m(n)$ from different prospective, for instance

$$\mathbf{P}_b^m(n) = \sum_{t=0}^{m} (-1)^{m-t} \mathbf{X}_{m,t}(b) \cdot n^t = \sum_{t=0}^{m} \sum_{\ell=1}^{2m-t+1} (-1)^{2m-t} \mathbf{H}_{m,t}(\ell) \cdot b^\ell \cdot n^t \tag{2.2}$$

The last line of the expression (2.2) clearly states why $\mathbf{P}_b^m(n)$ are polynomials in n, b. Let's show a few examples of polynomials $\mathbf{P}_b^m(n)$

$$\begin{aligned}
\mathbf{P}_b^0(n) &= +b, \\
\mathbf{P}_b^1(n) &= -2b^3 + 3b^2n + 3b^2 - 3bn, \\
\mathbf{P}_b^2(n) &= +6b^5 - 15b^4n - 15b^4 + 10b^3n^2 + 30b^3n + 10b^3 - 15b^2n^2 - 15b^2n + 5bn^2, \\
\mathbf{P}_b^3(n) &= -20b^7 + 70b^6n + 70b^6 - 84b^5n^2 - 210b^5n - 70b^5 + 35b^4n^3 + 210b^4n^2 + 175b^4n \\
&\quad - 70b^3n^3 - 140b^3n^2 + 28b^3 + 35b^2n^3 - 42b^2n - 7b^2 + 14bn^2 + 7bn, \\
\mathbf{P}_b^4(n) &= +70b^9 - 315b^8n - 315b^8 + 540b^7n^2 + 1260b^7n + 420b^7 - 420b^6n^3 - 1890b^6n^2 \\
&\quad - 1470b^6n + 126b^5n^4 + 1260b^5n^3 + 1890b^5n^2 - 294b^5 - 315b^4n^4 - 1050b^4n^3 \\
&\quad + 735b^4n + 210b^3n^4 - 630b^3n^2 + 180b^3 + 210b^2n^3 - 270b^2n - 60b^2 - 21bn^4 \\
&\quad + 90bn^2 + 60bn
\end{aligned}$$

We consider the polynomials $\mathbf{P}_b^m(n)$ because thanks to them we reveal the main aim of the work and establish a connection between the Binomial theorem and the discrete convolution of a power functions $\langle x\rangle^n$, $\{x\}^n$. In the next section we establish a relation between the polynomials $\mathbf{P}_b^m(n)$ and a power function of odd exponent $2m + 1, \quad m \geq 0$.

3. Relation between $\mathbf{P}_b^m(n)$ and power function of odd exponent

For instance, the odd powers $n^{2m+1}, \quad (n, m) \in \mathbb{N}$ are

$$\begin{aligned}
n^{2m+1} = \mathbf{P}_n^m(n) &= \sum_{r=0}^{m} \mathbf{A}_{m,r} \mathbf{C}_n^r(n) = \sum_{t=0}^{m} \sum_{\ell=1}^{2m-t+1} (-1)^{2m-t} \mathbf{H}_{m,t}(\ell) \cdot n^{\ell+t} \\
&= \sum_{t=0}^{m} (-1)^{m-t} \mathbf{X}_{m,t}(n) \cdot n^t
\end{aligned} \tag{3.1}$$

This relation could be also described in terms of limits as follows

$$\lim_{b \to n} \mathbf{P}_b^m(n) = n^{2m+1}$$

By the symmetry of $\mathbf{L}_m(n,k)$ the odd power $n^{2m+1}, \quad m \geq 0$ can be expressed as

$$n^{2m+1} = [n \text{ is even}]\mathbf{L}_m(n, n/2) + 2 \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \mathbf{L}_m(n,k),$$

where $[n \text{ is even}]$ is Iverson's bracket. Moreover, the Binomial expansion of $(x+y)^{2m+1}$ can be reached similarly

$$\begin{aligned}
(x+y)^{2m+1} &= \sum_{r=0}^{2m+1} \binom{2m+1}{r} x^{2m+1-r} y^r = \mathbf{P}^m_{x+y}(x+y) \\
&= \sum_{r=0}^{m} \mathbf{A}_{m,r} \mathbf{C}^r_{x+y}(x+y) \\
&= \sum_{r=0}^{m} \sum_{\ell=1}^{2m-r+1} (-1)^{2m-r} \mathbf{H}_{m,r}(\ell) \cdot (x+y)^{\ell+r} \\
&= \sum_{r=0}^{m} (-1)^{m-r} \mathbf{X}_{m,r}(x+y) \cdot (x+y)^r
\end{aligned} \tag{3.2}$$

It clearly follows that Multinomial expansion of odd-powered t-fold sum $(x_1+x_2+\cdots+x_t)^{2m+1}$ can be reached by $\mathbf{P}^m_b(x_1+x_2+\cdots+x_t)$ as well

(3.3)

$$\begin{aligned}
(x_1+x_2+\cdots+x_t)^{2m+1} &= \sum_{k_1+k_2+\cdots+k_t=2m+1} \binom{2m+1}{k_1, k_2, \ldots, k_t} \prod_{s=1}^{t} x_t^{k_s} \\
&= \mathbf{P}^m_{x_1+x_2+\cdots+x_t}(x_1+x_2+\cdots+x_t) \\
&= \sum_{r=0}^{m} \mathbf{A}_{m,r} \mathbf{C}^r_{x_1+x_2+\cdots+x_t}(x_1+x_2+\cdots+x_t) \\
&= \sum_{r=0}^{m} \sum_{\ell=1}^{2m-r+1} (-1)^{2m-r} \mathbf{H}_{m,r}(\ell) \cdot (x_1+x_2+\cdots+x_t)^{\ell+r} \\
&= \sum_{r=0}^{m} (-1)^{m-r} \mathbf{X}_{m,r}(x_1+x_2+\cdots+x_t) \cdot (x_1+x_2+\cdots+x_t)^r
\end{aligned}$$

4. Relation between $\mathbf{P}^m_b(n)$ and power function of natural exponent

In the previous section we have established a connection between the polynomial $\mathbf{P}^m_b(n)$ and a power function. As we can see, a particular case of the polynomial for $b=n$, which is identical to a power function of an odd exponent cannot be adapted for the general case of a power function without some changes. To go over to the general case of a power function, we consider the relationship between an even and an odd degree, namely

$$x^{\text{even}} = x \cdot x^{\text{odd}}$$

Thus, using the Iverson's bracket, a generalized power function can be expressed in terms of an odd power function as follows

$$x^s = x^{[s \text{ is even}]} \cdot x^{2\cdot\lfloor\frac{s-1}{2}\rfloor+1}$$

Thus, it is easy to generalise the partial case of $\mathbf{P}_b^m(n)$ for $b = n$ for all exponents $s \geq 1, \quad s \in \mathbb{N}$

$$\begin{aligned} x^s &= x^{[s \text{ is even}]}\mathbf{P}_x^h(x) = x^{[s \text{ is even}]}\sum_{k=0}^{h}\mathbf{A}_{h,k}\mathbf{C}_x^k(x) \\ &= x^{[s \text{ is even}]}\sum_{k=0}^{h}\sum_{\ell=1}^{2h-k+1}(-1)^{2h-k}\mathbf{H}_{h,k}(\ell)\cdot x^{\ell+k} \\ &= x^{[s \text{ is even}]}\sum_{k=0}^{h}(-1)^{h-k}\mathbf{X}_{h,k}(x)\cdot x^k, \end{aligned} \tag{4.1}$$

where $h = \lfloor\frac{s-1}{2}\rfloor$. The binomial expansion of $(x+y)^s$ for every $s \geq 1, \quad s \in \mathbb{N}$ is

$$\begin{aligned} (x+y)^s = \sum_{k=0}^{s}\binom{s}{k}x^{s-k}y^k &= (x+y)^{[s \text{ is even}]}\mathbf{P}_{x+y}^h(x+y) \\ &= (x+y)^{[s \text{ is even}]}\sum_{k=0}^{h}\mathbf{A}_{h,k}\mathbf{C}_{x+y}^k(x+y) \\ &= (x+y)^{[s \text{ is even}]}\sum_{k=0}^{h}\sum_{\ell=1}^{2h-k+1}(-1)^{2h-k}\mathbf{H}_{h,k}(\ell)\cdot(x+y)^{\ell+k} \\ &= (x+y)^{[s \text{ is even}]}\sum_{k=0}^{h}(-1)^{h-k}\mathbf{X}_{h,k}(x+y)\cdot(x+y)^k, \end{aligned} \tag{4.2}$$

where $h = \lfloor\frac{s-1}{2}\rfloor$. Now we are able to generalise the expression (4.1) for multinomial case as well. For the t-fold s-powered sum $(x_1+x_2+\cdots+x_t)^s, \quad s \geq 1$ we have following Multinomial expansion

$$(x_1 + x_2 + \cdots + x_t)^s = \sum_{k_1+k_2+\cdots+k_t=s} \binom{s}{k_1, k_2, \ldots, k_t} \prod_{\ell=1}^{t} x_\ell^{k_\ell}$$

$$= (x_1 + x_2 + \cdots + x_t)^{[s \text{ is even}]} \mathbf{P}^h_{x_1+x_2+\cdots+x_t}(x_1 + x_2 + \cdots + x_t)$$

$$= (x_1 + x_2 + \cdots + x_t)^{[s \text{ is even}]} \sum_{k=0}^{h} \mathbf{A}_{h,k} \mathbf{C}^k_{x_1+x_2+\cdots+x_t}(x_1 + x_2 + \cdots + x_t)$$

$$= (x_1 + x_2 + \cdots + x_t)^{[s \text{ is even}]} \sum_{k=0}^{h} \sum_{\ell=1}^{2h-k+1} (-1)^{2h-k} \mathbf{H}_{h,k}(\ell) \cdot (x_1 + x_2 + \cdots + x_t)^{\ell+t}$$

$$= (x_1 + x_2 + \cdots + x_t)^{[s \text{ is even}]} \sum_{k=0}^{h} (-1)^{h-k} \mathbf{X}_{h,k}(x_1 + x_2 + \cdots + x_t) \cdot (x_1 + x_2 + \cdots + x_t)^k,$$

where $h = \lfloor \frac{s-1}{2} \rfloor$.

5. Relations between the $\mathbf{P}^m_b(n)$ and discrete convolution of power functions $\langle x \rangle^n$, $\{x\}^n$

Previously we have established a relation between the polynomials $\mathbf{P}^m_b(n)$ and Binomial theorem. In this section a relation between $\mathbf{P}^m_b(n)$ and convolution of the power function $\langle x \rangle^n$ is established. To show that $\mathbf{P}^m_b(n)$ implicitly involves the discrete convolution of the power function $\langle x \rangle^n$ let's refresh what $\mathbf{P}^m_b(n)$ is

$$\mathbf{P}^m_b(n) = \sum_{r=0}^{m} \mathbf{A}_{m,r} \mathbf{C}^r_n(b)$$

Meanwhile, the term $\mathbf{C}^r_n(b)$ is the convolutional power sum of the form

$$\mathbf{C}^r_n(b) = \sum_{k=0}^{b-1} k^r (n-k)^r$$

By definition, the discrete convolution of defined over set of integers $\mathbb{Z}$ function f is

$$(f * f)[n] = \sum_{k=-\infty}^{\infty} f(k) f(n-k)$$

Now it could be noticed immediately that the discrete convolution $\langle x \rangle^n * \langle x \rangle^n$ of power function $\langle x \rangle^n$ is partial case of $\mathbf{C}^r_n(x)$. For instance, the discrete convolution of power functions $\langle x \rangle^n$ is

$$\langle x-a \rangle^n * \langle x-a \rangle^n = \sum_k \langle k+a \rangle^r \langle x-a-k \rangle^r = \sum_k (k+a)^r (x-a-k)^r [k \geq 0][x-k \geq 0]$$
$$= \sum_k (k+a)^r (x-a-k)^r [0 \leq k \leq x],$$

where $[0 \leq k \leq x]$ is Iverson bracket of k. In above equation we have applied a Knuth's recommendation [Knu92] concerning the sigma notation of sums. Thus, we have an identity between the discrete convolution $\langle x \rangle^n * \langle x \rangle^n$ of piecewise defined power function $\langle x \rangle^n$ and power sum $\mathbf{C}^r_n(x)$ for every $x \geq 1, \quad x \in \mathbb{N}$

$$\langle x-a \rangle^n * \langle x-a \rangle^n = \mathbf{C}^x_{x+1}(x-a)$$

For the case of discrete convolution of power function $\{n\}^r$ we have

$$\begin{aligned}\{x-a\}^n * \{x-a\}^n &= \sum_k \{k+a\}^r \{x-a-k\}^r = \sum_k (k+a)^r (x-a-k)^r [k>0][x-k>0] \\ &= \sum_k (k+a)^r (x-a-k)^r [0<k<x]\end{aligned}$$

Therefore, the polynomials $\mathbf{P}^m_b(n)$ are in relation with discrete convolutions $\langle x \rangle^n * \langle x \rangle^n$, $\{x\}^n * \{x\}^n$. For each $x \geq 1, \quad x \in \mathbb{N}$

$$\begin{aligned}&\mathbf{P}^m_x(x) = -1 + \sum_{r=0}^{m} \mathbf{A}_{m,r} \mathbf{C}^r_{x+1}(x) = -1 + \sum_{r=0}^{m} \mathbf{A}_{m,r} \langle x \rangle^r * \langle x \rangle^r = x^{2m+1} \\ &\mathbf{P}^m_x(x) = 1 + \sum_{r=0}^{m} \mathbf{A}_{m,r} \sum_{k=1}^{x-1} k^r (x-k)^r = 1 + \sum_{r=0}^{m} \mathbf{A}_{m,r} \{x\}^r * \{x\}^r = x^{2m+1}\end{aligned} \tag{5.1}$$

Note that in above formula the parameter a of power functions of $\langle x-a \rangle^r$, $\{x-a\}^r$ equals to $a=0$ and is not shown. By the identity $x^{2m+1} - 1 = \sum_{r=0}^{m} \mathbf{A}_{m,r} \{x\}^r * \{x\}^r$, the following property concerning the sum of $\mathbf{A}_{m,r}$, namely, the sum of the $\mathbf{A}_{m,r}$ equals to

$$\sum_{r=0}^{m} \mathbf{A}_{m,r} = 2^{2m+1} - 1$$

Above identity holds since the convolution $\{2\}^r * \{2\}^r = 1$ for each r. Furthermore, we are able to find a relation between the Binomial theorem and discrete convolution of power function $\langle x \rangle^n$.

6. Relation between Binomial theorem and discrete convolutions of power functions $\langle x \rangle^n$, $\{x\}^n$

As it is stated previously in (5.1), the polynomials $\mathbf{P}^m_b(n)$ are able to be expressed in terms of discrete convolution $\langle x \rangle^n * \langle x \rangle^n$ of the piecewise defined power function $\langle x \rangle^n$. In this section we refer to the power functions $\langle x \rangle^n$, $\{x\}^n$ and assume that parameter a to be equal $a=0$. By the equivalence between Binomial theorem and partial case of $\mathbf{P}^m_b(n)$, which is shown in (4.2), the odd Binomial expansion could be expressed in terms of discrete convolution of power function $\mathbf{P}^m_b(n)$ as well. For each $m \geq 0, \quad a+b \geq 1$ holds

$$\begin{aligned}(x+y)^{2m+1} &= \sum_{r=0}^{2m+1} \binom{2m+1}{r} x^{2m+1-r} y^r = -1 + \mathbf{P}^m_{x+y+1}(x+y) \\ &= -1 + \sum_{r=0}^{m} \mathbf{A}_{m,r} \mathbf{C}^r_{x+y+1}(x+y) = -1 + \sum_{r=0}^{m} \mathbf{A}_{m,r} \langle x+y \rangle^r * \langle x+y \rangle^r\end{aligned} \tag{6.1}$$

Above expression could be expressed in terms of discrete convolution $\{x\}^n * \{x\}^n$ as well,

$$(x+y)^{2m+1} = \sum_{r=0}^{2m+1} \binom{2m+1}{r} x^{2m+1-r} y^r = 1 + \sum_{r=0}^{m} \mathbf{A}_{m,r} \{x+y\}^r * \{x+y\}^r \tag{6.2}$$

In case of variations of a in $\langle x-a \rangle^n$ and $\{x-a\}^n$ the following identity holds for each $x \geq 2a, \quad a = const$

$$\begin{aligned}(x-2a)^{2m+1} &= -1 + \sum_{r=0}^{m} \mathbf{A}_{m,r} \langle x-a \rangle^r * \langle x-a \rangle^r \\ &= 1 + \sum_{r=0}^{m} \mathbf{A}_{m,r} \{x-a\}^r * \{x-a\}^r\end{aligned}$$

Note that a is parameter of $\langle x-a \rangle^r$, $\{x-a\}^r$ by definition (1.1). For the exponent $s \in \mathbb{N}$ we have the following relation between Binomial theorem and discrete convolution of $\langle x+y \rangle^s$. For each $s \geq 1, \quad x+y \geq 1$

$$\begin{aligned}(x+y)^s = \sum_{k=0}^{s} \binom{s}{k} x^{s-k} y^k &= (x+y)^{[s \text{ is even}]} \left(-1 + \mathbf{P}^h_{x+y+1}(x+y)\right) \\ &= (x+y)^{[s \text{ is even}]} \left(-1 + \sum_{r=0}^{h} \mathbf{A}_{h,r} \mathbf{C}^r_{x+y+1}(x+y)\right) \\ &= (x+y)^{[s \text{ is even}]} \left(-1 + \sum_{r=0}^{h} \mathbf{A}_{h,r} \langle x+y \rangle^r * \langle x+y \rangle^r\right),\end{aligned} \tag{6.3}$$

where $h = \lfloor \frac{s-1}{2} \rfloor$. In terms of convolution $\{x\}^n * \{x\}^n$ for $s \geq 1, \quad x+y \geq 2$ we also have the following relation

$$\begin{aligned}(x+y)^s = \sum_{k=0}^{s} \binom{s}{k} x^{s-k} y^k &= (x+y)^{[s \text{ is even}]} \left(1 + \mathbf{P}^h_{x+y}(x+y)\right) \\ &= (x+y)^{[s \text{ is even}]} \left(1 + \sum_{r=0}^{h} \mathbf{A}_{h,r} \mathbf{C}^r_{x+y}(x+y)\right) \\ &= (x+y)^{[s \text{ is even}]} \left(1 + \sum_{r=0}^{h} \mathbf{A}_{h,r} \{x+y\}^r * \{x+y\}^r\right),\end{aligned} \tag{6.4}$$

where $h = \lfloor \frac{s-1}{2} \rfloor$.

6.1. **Relations between Multinomial theorem and discrete convolutions of power functions $\langle x \rangle^n$, $\{x\}^n$.** In this subsection we'd like to generalise the equations (6.1), (6.2) to Mulitnomial case. For the t-fold s-powered sum $(x_1 + x_2 + \cdots + x_t)^s, \quad s \in \mathbb{N}, \quad x_1 + x_2 + \cdots + x_t \geq 1$ we have following equality involving Multinomial expansion and discrete convolution of power function $\langle x \rangle^n$. For each $s \geq 1,\ x_1 + x_2 + \cdots + x_t \geq 1$ holds

$$(x_1 + x_2 + \cdots + x_t)^s = \sum_{k_1+k_2+\cdots+k_t=s} \binom{s}{k_1, k_2, \ldots, k_t} \prod_{\ell=1}^{t} x_\ell^{k_\ell}$$

$$= (x_1 + x_2 + \cdots + x_t)^{[s \text{ is even}]} \left(-1 + \mathbf{P}^h_{x_1+x_2+\cdots+x_t}(x_1 + x_2 + \cdots + x_t)\right)$$

$$= (x_1 + x_2 + \cdots + x_t)^{[s \text{ is even}]} \left(-1 + \sum_{k=0}^{h} \mathbf{A}_{h,k} \mathbf{C}^k_{x_1+x_2+\cdots+x_t+1}(x_1 + x_2 + \cdots + x_t)\right)$$

$$= (x_1 + x_2 + \cdots + x_t)^{[s \text{ is even}]} \left(-1 + \sum_{k=0}^{h} \mathbf{A}_{h,k} \langle x_1 + x_2 + \cdots + x_t \rangle^k * \langle x_1 + x_2 + \cdots + x_t \rangle^k\right),$$

where $h = \lfloor \frac{s-1}{2} \rfloor$. In terms of convolution $\{x\}^n * \{x\}^n$ for $s \geq 1, \quad a + b \geq 2$ we also have the following relation

$$(x_1 + x_2 + \cdots + x_t)^s = \sum_{k_1+k_2+\cdots+k_t=s} \binom{s}{k_1, k_2, \ldots, k_t} \prod_{\ell=1}^{t} x_\ell^{k_\ell}$$

$$= (x_1 + x_2 + \cdots + x_t)^{[s \text{ is even}]} \left(1 + \sum_{k=0}^{h} \mathbf{A}_{h,k} \{x_1 + x_2 + \cdots + x_t\}^k * \{x_1 + x_2 + \cdots + x_t\}^k\right),$$

where $h = \lfloor \frac{s-1}{2} \rfloor$.

7. Power function as a product of certain matrices

8. Derivation of the coefficients $\mathbf{A}_{m,r}$

Assuming that for every integer $m \geq 0$

$$n^{2m+1} = \mathbf{P}^m_n(n) = \sum_{r=0}^{m} \mathbf{A}_{m,r} \mathbf{C}^r_n(n)$$

The coefficients $\mathbf{A}_{m,r}$ could be evaluated expanding $\mathbf{C}^r_n(n)$

$$\mathbf{C}^r_n(n) = \sum_{k=0}^{n-1} k^r (n-k)^r = \sum_{k=0}^{n-1} k^r \sum_{j=0}^{r} (-1)^j \binom{r}{j} n^{r-j} k^j = \sum_{j=0}^{r} (-1)^j \binom{r}{j} n^{r-j} \sum_{k=0}^{n-1} k^{r+j}$$

Using Faulhaber's formula $\sum_{k=1}^{n} k^p = \frac{1}{p+1}\sum_{j=0}^{p}\binom{p+1}{j}B_j n^{p+1-j}$, we get

(8.1)

$$\begin{aligned}\mathbf{C}_n^r(n) &= \sum_j \binom{r}{j} n^{r-j}\frac{(-1)^j}{r+j+1}\left[\sum_s \binom{r+j+1}{s} B_s n^{r+j+1-s} - B_{r+j+1}\right]\\ &= \sum_{j,s}\binom{r}{j}\frac{(-1)^j}{r+j+1}\binom{r+j+1}{s}B_s n^{2r+1-s} - \sum_j \binom{r}{j}\frac{(-1)^j}{r+j+1}B_{r+j+1}n^{r-j}\\ &= \sum_s \underbrace{\sum_j \binom{r}{j}\frac{(-1)^j}{r+j+1}\binom{r+j+1}{s}}_{S(r)} B_s n^{2r+1-s} - \sum_j \binom{r}{j}\frac{(-1)^j}{r+j+1}B_{r+j+1}n^{r-j}\end{aligned}$$

where B_s are Bernoulli numbers and $B_1 = \frac{1}{2}$. Now, we notice that

$$S(r) = \sum_j \binom{r}{j}\frac{(-1)^j}{r+j+1}\binom{r+j+1}{s} = \begin{cases}\frac{1}{(2r+1)\binom{2r}{r}}, & \text{if } s = 0;\\ \frac{(-1)^r}{s}\binom{r}{2r-s+1}, & \text{if } s > 0.\end{cases}$$

In particular, the last sum is zero for $0 < s \le r$. Therefore, expression (8.1) takes the form

$$\mathbf{C}_n^r(n) = \frac{1}{(2r+1)\binom{2r}{r}}n^{2r+1} + \underbrace{\sum_{s\ge1}\frac{(-1)^r}{s}\binom{r}{2r-s+1}B_s n^{2r+1-s}}_{(\star)}$$

$$-\underbrace{\sum_j \binom{r}{j}\frac{(-1)^j}{r+j+1}B_{r+j+1}n^{r-j}}_{(\diamond)}$$

Hence, introducing $\ell = 2r+1-s$ to $(\star)$ and $\ell = r-j$ to $(\diamond)$, we get

$$\begin{aligned}\mathbf{C}_n^r(n) &= \frac{1}{(2r+1)\binom{2r}{r}}n^{2r+1} + \sum_\ell \frac{(-1)^r}{2r+1-\ell}\binom{r}{\ell}B_{2r+1-\ell}n^\ell\\ &\quad - \sum_\ell \binom{r}{\ell}\frac{(-1)^{j-\ell}}{2r+1-\ell}B_{2r+1-\ell}n^\ell\\ &= \frac{1}{(2r+1)\binom{2r}{r}}n^{2r+1} + 2\sum_{\text{odd } \ell}\frac{(-1)^r}{2r+1-\ell}\binom{r}{\ell}B_{2r+1-\ell}n^\ell\end{aligned}$$

Using the definition of $\mathbf{A}_{m,r}$ coefficients, we obtain the following identity for polynomials in n

$$\sum_r \mathbf{A}_{m,r}\frac{1}{(2r+1)\binom{2r}{r}}n^{2r+1} + 2\sum_{r,\ \text{odd } \ell}\mathbf{A}_{m,r}\frac{(-1)^r}{2r+1-\ell}\binom{r}{\ell}B_{2r+1-\ell}n^\ell \equiv n^{2m+1} \tag{8.2}$$

Taking the coefficient of n^{2m+1} in (8.2) we get $\mathbf{A}_{m,m} = (2m+1)\binom{2m}{m}$ and taking the coefficient of n^{2d+1} for an integer d in the range $m/2 \le d < m$, we get $\mathbf{A}_{m,d} = 0$. Taking the coefficient of n^{2d+1} for d in the range $m/4 \le d < m/2$, we get

$$\mathbf{A}_{m,d}\frac{1}{(2d+1)\binom{2d}{d}} + 2(2m+1)\binom{2m}{m}\binom{m}{2d+1}\frac{(-1)^m}{2m-2d}B_{2m-2d} = 0,$$

i.e,

$$\mathbf{A}_{m,d} = (-1)^{m-1} \frac{(2m+1)!}{d!d!m!(m-2d-1)!} \frac{1}{m-d} B_{2m-2d}.$$

Continue similarly, we can express $\mathbf{A}_{m,d}$ for each integer d in range $m/2^{s+1} \leq d < m/2^s$ (iterating consecutively $s = 1, 2...$) via previously determined values of $\mathbf{A}_{m,j}$ as follows

$$\mathbf{A}_{m,d} = (2d+1)\binom{2d}{d} \sum_{j \geq 2d+1} \mathbf{A}_{m,j} \binom{j}{2d+1} \frac{(-1)^{j-1}}{j-d} B_{2j-2d}.$$

9. Verification of the results and examples

To fulfill our study we provide an opportunity to verify its results by means of Wolfram Mathematica language. It is possible to verify the most important results of the manuscript using the Mathematica programs available at https://github.com/kolosovpetro/research_unit_tests. Also, we'd like to show why an odd-power identity (3.1) holds by a few examples. We arrange in tables the values of $\mathbf{L}_m(n,k)$ to show that $\mathbf{P}_n^m(n) = \mathbf{L}_m(n,0) + \mathbf{L}_m(n,1) + \cdots + \mathbf{L}_m(n,n-1) = n^{2m+1}$. For example, for $m = 1$ we have the following values of $\mathbf{L}_1(n,k)$

n/k	0	1	2	3	4	5	6	7
0	1							
1	1	1						
2	1	7	1					
3	1	13	13	1				
4	1	19	25	19	1			
5	1	25	37	37	25	1		
6	1	31	49	55	49	31	1	
7	1	37	61	73	73	61	37	1

Table 2. Triangle generated by $\mathbf{L}_1(n,k),\ 0 \leq k \leq n$.

From table 1 it is seen that

$$\begin{aligned}
\mathbf{P}_0^1(0) &= 0 = 0^3 \\
\mathbf{P}_1^1(1) &= 1 = 1^3 \\
\mathbf{P}_2^1(2) &= 1 + 7 = 2^3 \\
\mathbf{P}_3^1(3) &= 1 + 13 + 13 = 3^3 \\
\mathbf{P}_4^1(4) &= 1 + 19 + 25 + 19 = 4^3 \\
\mathbf{P}_5^1(5) &= 1 + 25 + 37 + 37 + 25 = 5^3.
\end{aligned}$$

Another case, if $m = 2$, we have the following values of $\mathbf{L}_2(n,k)$

n/k	0	1	2	3	4	5	6	7
0	1							
1	1	1						
2	1	31	1					
3	1	121	121	1				
4	1	271	481	271	1			
5	1	481	1081	1081	481	1		
6	1	751	1921	2431	1921	751	1	
7	1	1081	3001	4321	4321	3001	1081	1

Table 3. Triangle generated by $\mathbf{L}_2(n,k),\ 0 \leq k \leq n$.

Again, an odd-power identity (3.1) holds

$$\mathbf{P}^2_0(0) = 0 = 0^5$$
$$\mathbf{P}^2_1(1) = 1 = 1^5$$
$$\mathbf{P}^2_2(2) = 1 + 31 = 2^5$$
$$\mathbf{P}^2_3(3) = 1 + 121 + 121 = 3^5$$
$$\mathbf{P}^2_4(4) = 1 + 271 + 481 + 271 = 4^5$$
$$\mathbf{P}^2_5(5) = 1 + 481 + 1081 + 1081 + 481 = 5^5.$$

Tables (1), (2) are entries A287326, A300656 in [Slo64].

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11. Conclusion

In this manuscript a relation between Binomial theorem and discrete convolution of piecewise defined power function is established. It is shown that Binomial expansion of s-powered sum $(a+b)^s,\ s \geq 1$ is equivalent to the sum of consequent convolutions of piecewise defined power function multiplied by the certain real coefficients. In addition, the relation between Binomial theorem and convolution of piecewise defined power function is generalised to Multinomial case.

References

[AS72] Milton Abramowitz and Irene A. Stegun, editors. *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables.* U.S. Government Printing Office, Washington, DC, USA, tenth printing edition, 1972.

[BDM11] Steven B. Damelin and Willard Miller. The mathematics of signal processing. *The Mathematics of Signal Processing*, page 232, 01 2011.

[GKP94] Ronald L. Graham, Donald E. Knuth, and Oren Patashnik. *"Binomial Coefficients" in Concrete Mathematics: A Foundation for Computer Science*. Addison-Wesley Longman Publishing Co., Inc., Boston, MA, USA, 2nd edition, 1994.

[Ive62] Kenneth E. Iverson. A programming language. In *Proceedings of the May 1-3, 1962, Spring Joint Computer Conference*, AIEE-IRE '62 (Spring), pages 345–351, New York, NY, USA, 1962. ACM.

[Knu92] Donald E. Knuth. Two notes on notation. *Am. Math. Monthly*, 99(5):403–422, 1992.

[Mac19] W. H. Macaulay. A note on the deflection of beams. *Messenger of Mathematics*, 48:129, 1919.

[Slo64] N. J. A. Sloane. The on-line encyclopedia of integer sequences. published electronically at https://oeis.org, 1964.

[Wei] Eric W Weisstein. "Bernoulli Number." From MathWorld – A Wolfram Web Resource. http://mathworld.wolfram.com/BernoulliNumber.html.

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