FAULHABER'S COEFFICIENTS: EXAMPLES

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ABSTRACT. Examples of Faulhaber's coefficients as per Johann Faulhaber and sums of powers [1].

1. Introduction

The work Johann Faulhaber and sums of powers [1, p. 16] provides the following identity for sums of odd powers

$$\Sigma n^{2m-1} = \frac{1}{2m}(B_{2m}(n+1) - B_{2m}) = \frac{1}{2m}(A_0^{(m)} u^m + A_1^{(m)} u^{m-1} + \cdots A_{m-1}^{(m)} u)$$

where $A_r^{(m)}$ are Faulhaber's coefficients, and $u = n^2 + n$. For every $r > m$ or $r < 0$ the coefficients $A_r^{(m)}$ are zeroes. In Knuth's notation, the sigma Σn^{2m-1} denotes the sum of powers $\Sigma n^{2m-1} = 1^{2m-1} + 2^{2m-1} + \cdots n^{2m-1}$. Consider the equation above with the summation limits defined explicitly

$$\sum_{k=1}^{p} k^{2m-1} = \frac{1}{2m}(A_0^{(m)} u^m + A_1^{(m)} u^{m-1} + \cdots A_{m-1}^{(m)} u)$$

where $u = p^2 + p$. As expected, the power sum $\sum_{k=1}^{p} k^{2m+1}$ has a closed form polynomial in p, which corresponds to Faulhaber's formula. The coefficients $A_r^{(m)}$ are defined by

$$A_k^{(m)} = \begin{cases} B_{2m} & \text{if } k = m \\ (-1)^{m-k} \sum_j \binom{2m}{m-k-j} \binom{m-k+j}{j} \frac{m-k-j}{m-k+j} B_{m+k+j} & \text{if } 0 \leq k < m \\ 0 & \text{if } r < 0 \text{ or } r > m \end{cases}$$

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For example,

m/k	0	1	2	3	4	5	6	7	8	9	10
0	1										
1	1	$\frac{1}{6}$									
2	1	0	$-\frac{1}{30}$								
3	1	$-\frac{1}{2}$	0	$\frac{1}{42}$							
4	1	$-\frac{4}{3}$	$\frac{2}{3}$	0	$-\frac{1}{30}$						
5	1	$-\frac{5}{2}$	3	$-\frac{3}{2}$	0	$\frac{5}{66}$					
6	1	-4	$\frac{17}{2}$	-10	5	0	$-\frac{691}{2730}$				
7	1	$-\frac{35}{6}$	$\frac{287}{15}$	$-\frac{118}{3}$	$\frac{691}{15}$	$-\frac{691}{30}$	0	$\frac{7}{6}$			
8	1	-8	$\frac{112}{3}$	$-\frac{352}{3}$	$\frac{718}{3}$	-280	140	0	$-\frac{3617}{510}$		
9	1	$-\frac{21}{2}$	66	-293	$\frac{4557}{5}$	$-\frac{3711}{2}$	$\frac{10851}{5}$	$-\frac{10851}{10}$	0	$\frac{43867}{798}$	
10	1	$-\frac{40}{3}$	$\frac{217}{2}$	$-\frac{4516}{7}$	2829	$-\frac{26332}{3}$	$\frac{750167}{42}$	$-\frac{438670}{21}$	$\frac{219335}{21}$	0	$-\frac{174611}{330}$

Table 1. Faulhaber's coefficients $A_k^{(m)}$.

In its explicit form the sum of odd powers is

$$\sum_{k=1}^{p} k^{2m-1} = \frac{1}{2m} \sum_{r=0}^{m-1} A_r^{(m)} (p^2+p)^{m-r}$$

Consider the examples of power sums for various values of m, while setting $u = p^2 + p$

$$\sum_{k=1}^{p} n = \frac{1}{2} u \qquad = \frac{1}{2} A_0^{(1)} u$$

$$\sum_{k=1}^{p} n^3 = \frac{1}{4} u^2 \qquad = \frac{1}{4}\left(A_0^{(2)} u^2 + A_1^{(2)} u\right)$$

$$\sum_{k=1}^{p} n^5 = \frac{1}{6}\left(u^3 - \frac{1}{2} u^2\right) \qquad = \frac{1}{6}\left(A_0^{(3)} u^3 + A_1^{(3)} u^2 + A_2^{(3)} u\right)$$

$$\sum_{k=1}^{p} n^7 = \frac{1}{8}\left(u^4 - \frac{4}{3} u^3 + \frac{2}{3} u^2\right) \qquad = \frac{1}{8}\left(A_0^{(4)} u^4 + A_1^{(4)} u^3 + A_2^{(4)} u^2 + A_3^{(4)} u\right)$$

$$\sum_{k=1}^{p} n = \frac{1}{2} \cdot 1 \cdot (p^2+p) = \frac{1}{2}\left(p^2+p\right)$$

$$\sum_{k=1}^{p} n^3 = \frac{1}{4}\left(1 \cdot (p^2+p)^2 + 0 \cdot (p^2+p)\right) = \frac{1}{4}\left(p^2+p\right)^2$$

$$\sum_{k=1}^{p} n^5 = \frac{1}{6}\left(1 \cdot (p^2+p)^3 - \frac{1}{2} \cdot (p^2+p)^2 + 0 \cdot (p^2+p)\right) = \frac{1}{6}\left(\left(p^2+p\right)^3 - \frac{1}{2}\left(p^2+p\right)^2\right)$$

$$\sum_{k=1}^{p} n^7 = \frac{1}{8}\left(1 \cdot (p^2+p)^4 - \frac{4}{3} \cdot (p^2+p)^3 + \frac{2}{3} \cdot (p^2+p)^2 + 0 \cdot (p^2+p)\right)$$
$$= \frac{1}{8}\left(\left(p^2+p\right)^4 - \frac{4}{3}\left(p^2+p\right)^3 + \frac{2}{3}\left(p^2+p\right)^2\right)$$

Mathematica functions to validate, see this `GitHub repository`

- `FaulhaberCoefficients[n,k]` validates the coefficients $A_r^{(m)}$
- `FaulhaberSum[p,m]` validates the identity $\sum_{k=1}^{p} k^{2m-1} = \frac{1}{2m}\sum_{r=0}^{m-1} A_r^{(m)}(p^2+p)^{m-r}$
- `SumOfOddPowers[p, m]` power sum $\sum_{k=1}^{p} k^{2m-1}$, the result matches with $\sum_{k=1}^{p} k^{2m-1} = \frac{1}{2m}\sum_{r=0}^{m-1} A_r^{(m)}(p^2+p)^{m-r}$

References

[1] Knuth, Donald E. Johann Faulhaber and sums of powers. *Mathematics of Computation*, 61(203):277–294, 1993. `https://arxiv.org/abs/math/9207222`.

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