# Sums of powers of positive integers and their recurrence relations

Luciano Ancora

# Contents

## 0.1 Second partial sums of m-th powers

We consider the sums of powers of successive integers:

$$\sum_{k=1}^{n} k^m = 1^m + 2^m + ... + n^m$$

which, as we know, are calculated with the Faulhaber polynomials, as follows:

$$\sum_{k=1}^{n} k = 1/2(n^2 + n)$$
$$\sum_{k=1}^{n} k^2 = 1/6(2n^3 + 3n^2 + n)$$
$$\sum_{k=1}^{n} k^3 = 1/4(n^4 + 2n^3 + n^2)$$
$$\sum_{k=1}^{n} k^4 = 1/30(6n^5 + 15n^4 + 10n^3 - n) \qquad \text{... etc. etc.}$$

Each of these formulas generates, as $n$ varies, an increasing numerical sequence, of the type of that obtained for $m = 2$:

$1, 5, 14, 30, 55, 91, 140, 204, 285, ...$

that is the sequence of the square pyramidal numbers .

We aim to find a way to calculate, given any of these sequences, the sum of its first $n$ terms, that is, the *second partial sums of the m-th powers.*

An opportunity to obtain this is offered by the following table:

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $1^m$ | $1^m$ | $1^m$ | $1^m$ | $1^m$ | $1^m$ | $1^m$ | $1^m$ | ... | $1^m$ | $1^m$ |
| $2^m$ | $2^m$ | $2^m$ | $2^m$ | $2^m$ | $2^m$ | $2^m$ | $2^m$ | ... | $2^m$ | $2^m$ |
| $3^m$ | $3^m$ | $3^m$ | $3^m$ | $3^m$ | $3^m$ | $3^m$ | $3^m$ | ... | $3^m$ | $3^m$ |
| $4^m$ | $4^m$ | $4^m$ | $4^m$ | $4^m$ | $4^m$ | $4^m$ | $4^m$ | ... | $4^m$ | $4^m$ |
| $5^m$ | $5^m$ | $5^m$ | $5^m$ | $5^m$ | $5^m$ | $5^m$ | $5^m$ | ... | $5^m$ | $5^m$ |
| $6^m$ | $6^m$ | $6^m$ | $6^m$ | $6^m$ | $6^m$ | $6^m$ | $6^m$ | ... | $6^m$ | $6^m$ |
| $7^m$ | $7^m$ | $7^m$ | $7^m$ | $7^m$ | $7^m$ | $7^m$ | $7^m$ | ... | $7^m$ | $7^m$ |
| : | : | : | : | : | : | : | : | : | : | : |
| $(n-1)^m$ | $(n-1)^m$ | $(n-1)^m$ | $(n-1)^m$ | $(n-1)^m$ | $(n-1)^m$ | $(n-1)^m$ | $(n-1)^m$ | ... | $(n-1)^m$ | $(n-1)^m$ |
| $n^m$ | $n^m$ | $n^m$ | $n^m$ | $n^m$ | $n^m$ | $n^m$ | $n^m$ | ... | $n^m$ | $n^m$ |

By summing the content of each column (green+yellow boxes), we obtain the sum of the first $n$ $m$-th powers, that (in tribute to Faulhaber) we denote by $F_{(m)}$ :

$$F_{(m)} = \sum_{k=1}^{n} k^m$$

and the contents of the entire table will be then:

$$(n+1)F_{(m)} = (n+1)\sum_{k=1}^{n} k^m$$

The green section contains, in each row, the amount: $K^m \times K = K^{(m+1)}$. By summing the contents of all rows of this section, one obtains the sum of the first $n$ $(m+1)$-th powers:

$$F_{(m+1)} = \sum_{k=1}^{n} k^{(m+1)}$$

The yellow section contains, in the columns, the sequence of $F_{(m)}$ sums. By summing the contents of all columns of this section, one obtains the second partial sums of the $m$-th powers.

The quantity that we seek is then obtained by subtracting to the content of the entire table, the content of the green boxes, that is:

$$\boxed{\sum_{k=1}^{n} F_{(m)} = (n+1)F_{(m)} - F_{(m+1)}} \tag{1}$$

As we shall see later, this formula will enable us to conduct a research, limitless and in every direction, to find polynomial expressions generating integer sequences of sums of powers of every order and grade.

By performing algebraic calculations for $m = 3$, you get:

$$\sum_{k=1}^{n} F_{(3)} = (n+1)F_{(3)} - F_{(4)} = (n+1)[(n^2+n)/2]^2 - (6n^5 + 15n^4 + 10n^3 - n)/30 = (3n^5 + 15n^4 + 25n^3 + 15n^2 + 2n)/60$$

Polynomials obtained for $m$ from 1 to 8 are listed in the following table:

| $m$ | Second partial sums of $m$-th powers |
|---|---|
| 1 | $(n^3 + 3n^2 + 2n)/6$ |
| 2 | $(n^4 + 4n^3 + 5n^2 + 2n)/12$ |
| 3 | $(3n^5 + 15n^4 + 25n^3 + 15n^2 + 2n)/60$ |
| 4 | $(2n^6 + 12n^5 + 25n^4 + 20n^3 + 3n^2 - 2n)/60$ |
| 5 | $(2n^7 + 14n^6 + 35n^5 + 35n^4 + 7n^3 - 7n^2 - 2n)/84$ |
| 6 | $(3n^8 + 24n^7 + 70n^6 + 84n^5 + 21n^4 - 28n^3 - 10n^2 + 4n)/168$ |
| 7 | $(5n^9 + 45n^8 + 150n^7 + 210n^6 + 63n^5 - 105n^4 - 50n^3 + 30n^2 + 12n)/360$ |
| 8 | $(2n^{10} + 20n^9 + 75n^8 + 120n^7 + 42n^6 - 84n^5 - 50n^4 + 40n^3 + 21n^2 - 6n)/180$ |
| | |

Polynomial expressions generated by (1) are the natural extension of those listed at the beginning. The general formula for obtaining them in a direct way is written, in the compact notation of Faulhaber formula, in the following way:

$$\sum_{k=1}^{n} F_m = (n+1)F_m - F_{(m+1)}$$
$$= \frac{n+1}{m+1}\sum_{k=0}^{m}(-1)^k \binom{m+1}{k} B_k n^{m+1-k} - \frac{1}{m+2}(-1)^k \sum_{k=0}^{m+1}\binom{m+2}{k} B_k (n+1)^{m+2-k}$$

where the $B_k$ quantities are the Bernoulli numbers .

This formula calculates, for each natural number $n$, the second partial sums of the $m$-th powers.

## 0.2 Partial sums of m-th powers with Faulhaber polynomials

In previous section we saw how to use Faulhaber polynomials $F_{(m)}$ to obtain formulas of second partial sums of $m$-th powers.

We will seek now an iterative process for deriving, starting from the formula (1) of previous section, the polynomial expressions that calculate the partial sums later.

Whereas each partial sum represents, by definition, the sum of the first $q$ terms of the previous sum, we have, in general:

$$P_{j(m)} = \sum_{n=1}^{q} P_{j-1(m)}$$

Then, the polynomial expression of the $j$-th partial sum of the $m$-th power is obtained by adding (from 1 to $q$) all terms of the polynomial of the previous partial sum. In the practice is better to use non factored polynomials, ranked according to $n$ powers: to each power of $n$ will correspond the $F_{(m)}$ polynomial.

We will perform the derivation of the polynomial $P_3$ on the first power:

$$P_{2(1)} = (n^3 + 3n^2 + 2n)/6$$

$$P_{3(1)} = \sum_{n=1}^{q} P_{2(1)} = \sum_{n=1}^{q} (n^3 + 3n^2 + 2n)/6$$
$$= (F_{(3)} + 3F_{(2)} + 2F_{(1)})/6 = n(1+n)(2+n)(3+n)/24$$

The knowledge [1] of the general formula (1) enables then, using Faulhaber polynomials, the direct derivation of the $m$-th power formulas.

With Mathematica , using mainly Factor and Table commands and a bit of copy and paste, you get faster:

| m | Third partial sums of m-th powers: $P_{3(m)}$ |
|---|---|
| 1 | n*(1 + n)*(2 + n)*(3 + n)/24 |
| 2 | n*(1 + n)*(2 + n)*(3 + n)*(3 + 2*n)/120 |
| 3 | n*(1 + n)*(2 + n)*(3 + n)*(1 + 3*n + n^2)/120 |
| 4 | n*(1 + n)*(2 + n)*(3 + n)*(3 + 2*n)*(-1 + 6*n +2* n^2)/840 |
| 5 | n*(1 + n)*(2 + n)*(3 + n)*(-1 + 2*n + n^2)*(2 + 4*n + n^2)/336 |
| 6 | n*(1 + n)*(2 + n)*(3 + n)*(3 + 2*n)*(2 - 30*n + 35*n^2 + 30*n^3 + 5*n^4)/5040 |
| 7 | n*(1 + n)*(2 + n)*(3 + n)*(6 - 6*n - 20*n^2 + 15*n^3 + 25*n^4 + 9*n^5 + n^6)/720 |
| 8 | n*(1 + n)*(2 + n)*(3 + n)*(3 + 2*n)*(1 + 36*n - 69*n^2 + 45*n^4 + 18*n^5 + 2*n^6)/3960 |
| | |

[1] Ignoring the (1) we would not have $P_{2(1)}$ that enabled us to derive $P_{3(1)}$.

| m | Fourth partial sums of m-th powers: $P_{4(m)}$ |
|---|---|
| 1 | n*(1 + n)*(2 + n)*(3 + n)*(4 + n)/120 |
| 2 | n*(1 + n)*(2 + n)^2*(3 + n)*(4 + n)/360 |
| 3 | n*(1 + n)*(2 + n)*(3 + n)*(4 + n)*(2 + 4*n + n^2)/840 |
| 4 | n*(1 + n)*(2 + n)^2*(3 + n)*(4 + n)*(-1 + 12*n + 3*n^2)/5040 |
| 5 | n*(1 + n)*(2 + n)*(3 + n)*(4 + n)*(-24 + 20*n + 85*n^2 + 40*n^3 + 5*n^4)/15120 |
| 6 | n*(1 + n)*(2 + n)^2*(3 + n)*(4 + n)*(-1 - 8*n + 14*n^2 + 8*n^3 + n^4)/5040 |
| 7 | n*(1 + n)*(2 + n)*(3 + n)*(4 + n)*(48 - 100*n - 89*n^2 + 160*n^3 + 140*n^4 + 36*n^5 + 3*n^6)/23760 |
| 8 | n*(1 + n)*(2 + n)^2*(3 + n)*(4 + n)*(1 + 4*n + n^2)*(21 - 48*n + 20*n^2 + 16*n^3 + 2*n^4)/23760 |
| | |

| m | Fifth partial sums of m-th powers: $P_{5(m)}$ |
|---|---|
| 1 | n*(1 + n)*(2 + n)*(3 + n)*(4 + n)*(5 + n)/720 |
| 2 | n*(1 + n)*(2 + n)*(3 + n)*(4 + n)*(5 + n)*(5 + 2*n)/5040 |
| 3 | n*(1 + n)*(2 + n)*(3 + n)*(4 + n)*(5 + n)*(10 + 15*n + 3*n^2)/20160 |
| 4 | n^2*(1 + n)*(2 + n)*(3 + n)*(4 + n)*(5 + n)^2*(5 + 2*n)/30240 |
| 5 | n*(1 + n)*(2 + n)*(3 + n)*(4 + n)*(5 + n)*(-2 + 5*n + n^2)*(9 + 10*n + 2*n^2)/60480 |
| 6 | n*(1 + n)*(2 + n)*(3 + n)*(4 + n)*(5 + n)*(5 + 2*n)*(-3 + 5*n + n^2)*(4 + 15*n + 3*n^2)/332640 |
| 7 | n*(1 + n)*(2 + n)*(3 + n)*(4 + n)*(5 + n)*(-3 + 5*n + n^2)*(-2 + 5*n + n^2)*(5 + 5*n + n^2)/95040 |
| | |

... etc. etc.

## 0.3 Pascal's triangle and recurrence relations for partial sums of m-th powers

We build with Excel the table that calculates, for successive additions (left cell+top cell) the partial sums of powers of positive integers:

| $n$ | $n^m$ | $1^{th}$ sums | $2^{th}$ sums | $3^{th}$ sums |
|---|---|---|---|---|
| 1 | ... | ... | ... | ... |
| 2 | ... | ... | ... | ... |
| 3 | ... | ... | d | ... |
| 4 | ... | b | e | ... |
| 5 | a | c | f | ... |
| 6 | ... | ... | ... | ... |
| 7 | ... | ... | ... | ... |

We want to obtain the recurrence relation for second sums, that is a formula for calculating the $n$-th term in the column "2th sums" as a function of the previous terms.

The formula that we seek is obtained by analyzing the data in the table as follows:

c = a + b
e = b + d
f = c + e = a + b + e = a + e - d + e
f = 2e - d + a

Indicating with $a_{(n,m)}$ the $n$-th term of the sequence, we therefore have:

2th sums: $a_{(n,m)} = 2a_{(n-1,m)} - a_{(n-2,m)} + n^m$

Extending the previous scheme to the successive columns, one obtains:

3th sums: $a_{(n,m)} = 3a_{(n-1,m)} - 3a_{(n-2,m)} + a_{(n-3,m)} + n^m$

4th sums: $a_{(n,m)} = 4a_{(n-1,m)} - 6a_{(n-2,m)} + 4a_{(n-3,m)} - a_{(n-4,m)} + n^m$

From this point forward we continue (successfully) using Pascal's triangle , by alternating signs, with the following results:

5th sums: $a_{(n,m)} = 5a_{(n-1,m)} - 10a_{(n-2,m)} + 10a_{(n-3,m)} - 5a_{(n-4,m)} + a_{(n-5,m)} + n^m$

and so on ...

So, if we denote by $p$ the order number of the partial sums, its recurrence relation is obtained by the following general formula:

$$a_{(n,m)} = \left[ \sum_{k=0}^{p-1} (-1)^k \binom{p}{k+1} a_{(n-1-k,m)} \right] + n^m$$

This was a "non linear" recurrence relation between the sequence terms. There are also "linear" recurrence relations, obtainable from Pascal’s triangle, having the following general formula:

$$a_{(n,m)} = \sum_{k=0}^{p+m-1} (-1)^k \binom{p+m}{k+1} a_{(n-1-k,m)}$$

Example: for the "fourth partial sums of sixth powers" you get the linear relationship:

$$\begin{aligned} a_{(n)} = {} & 10a_{(n-1)} - 45a_{(n-2)} + 120a_{(n-3)} - 210a_{(n-4)} + 252a_{(n-5)} - 210a_{(n-6)} \\ & +120a_{(n-7)} - 45a_{(n-8)} + 10a_{(n-9)} - a_{(n-10)} \end{aligned}$$

The coefficient lists of such expressions are shown in OEIS with the term "signature".

## 0.4 Horizontal sequences

We insert in Excel sequences of "Third partial sums of $m$-th powers", arranging them in a table as follows:

| Second partial sums of m-th powers | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $n$ | $m$=0 | $m$=1 | $m$=2 | $m$=3 | $m$=4 | $m$=5 | $m$=6 | $m$=7 |
| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 2 | 3 | 4 | 6 | 10 | 18 | 34 | 66 | 130 |
| 3 | 6 | 10 | 20 | 46 | 116 | 310 | 860 | 2446 |
| 4 | 10 | 20 | 50 | 146 | 470 | 1610 | 5750 | 21146 |
| 5 | 15 | 35 | 105 | 371 | 1449 | 6035 | 26265 | 117971 |
| 6 | 21 | 56 | 196 | 812 | 3724 | 18236 | 93436 | 494732 |
| 7 | 28 | 84 | 336 | 1596 | 8400 | 47244 | 278256 | 1695036 |
| 8 | 36 | 120 | 540 | 2892 | 17172 | 109020 | 725220 | 4992492 |
| 9 | 45 | 165 | 825 | 4917 | 32505 | 229845 | 1703625 | 13072917 |
| 10 | 55 | 220 | 1210 | 7942 | 57838 | 450670 | 3682030 | 31153342 |
| 11 | 66 | 286 | 1716 | 12298 | 97812 | 832546 | 7431996 | 68720938 |
| 12 | 78 | 364 | 2366 | 18382 | 158522 | 1463254 | 14167946 | 142120342 |

Consider the recurrence relation for 2th sums seen above:

$$a_{(n,m)} = 2a_{(n-1,m)} - a_{(n-2,m)} + n^m$$

We will use this relationship to derive formulas of sequences that appear in each row of the table.

The opportunity to do this lies in the fact that the above relationship is valid, for all $n$, in every column of the table, and then for each term of the considered horizontal sequence. You get formulas by replacing subsequently, in the general recurrence relation, polynomials that gradually you get, starting from n = 2. This is an inductive process that applies to any order of partial sums of $m$-th powers.

We perform here derivations of formulas until the $a_{(6,m)}$ sequence.

$n$=2: $a_{(2,m)} = 2 \times 1 - 0 + 2^m = \boxed{2^m + 2}$

$n$=3: $a_{(3,m)} = 2a_{(2,m)} - a_{(1,m)} + 3^m = 2(2^m + 2) - 1 + 3^m = \boxed{2^{m+1} + 3^m + 3}$

$n$=4: $a_{(4,m)} = 2a_{(3,m)} - a_{(2,m)} + 4^m = 2(2^{m+1} + 3^m + 3) - (2^m + 2) + 4^m$
$= \boxed{3 \times 2^m + 2^{2m} + 2 \times 3^m + 4}$

$n$=5: $a_{(5,m)} = \boxed{2^{m+2} + 2^{2m+1} + 3^{m+1} + 5^m + 5}$

$n$=6: $a_{(6,m)} = \boxed{5 \times 2^m + 3 \times 4^m + 4 \times 3^m + 2 \times 5^m + 6^m + 6}$

This process works indefinitely, generating polynomial expressions longer and longer, which in turn generate sequences with terms that magnify more and more rapidly. I personally found sequences up to the fifth partial sums, visible in the OEIS Encyclopedia to the above link.

## 0.5 The differences of m-th powers

The first differences of the $m$-th powers are computed, by definition, in the following way:

$$D_{1(m)} = n^m - (n-1)^m \tag{2}$$

Using the program "Mathematica" you get from the (2) the following complete polynomials:

| $m$ | First differences of $m$-th powers: $D_{1(m)} = n^m - (n-1)^m$ | |
|---|---|---|
| 1 | 1 | |
| 2 | -1 + 2 n | OFF. 1 |
| 3 | 1 - 3 n + 3 n^2 | |
| 4 | -1 + 4 n - 6 n^2 + 4 n^3 | |
| 5 | 1 - 5 n + 10 n^2 - 10 n^3 + 5 n^4 | |
| 6 | -1 + 6 n - 15 n^2 + 20 n^3 - 15 n^4 + 6 n^5 | |
| 7 | 1 - 7 n + 21 n^2 - 35 n^3 + 35 n^4 - 21 n^5 + 7 n^6 | |
| 8 | -1 + 8 n - 28 n^2 + 56 n^3 - 70 n^4 + 56 n^5 - 28 n^6 + 8 n^7 | |
| 9 | 1 - 9 n + 36 n^2 - 84 n^3 + 126 n^4 - 126 n^5 + 84 n^6 - 36 n^7 + 9 n^8 | |

You will get the subsequent differences by replacing $n$ with $(n-1)$ in the corresponding polynomials having degree >1 of the preceding tables:

| $m$ | Second differences of $m$-th powers: $D_{2(m)}$ | |
|---|---|---|
| 1 | | |
| 2 | 2 | OFF. 2 |
| 3 | 6 (-1 + n) | |
| 4 | 2 (7 - 12 n + 6 n^2) | |
| 5 | 10 (-3 + 7 n - 6 n^2 + 2 n^3) | |
| 6 | 2 (31 - 90 n + 105 n^2 - 60 n^3 + 15 n^4) | |
| 7 | 14 (-1 + n) (9 - 22 n + 23 n^2 - 12 n^3 + 3 n^4) | |
| 8 | 2 (127 - 504 n + 868 n^2 - 840 n^3 + 490 n^4 - 168 n^5 + 28 n^6) | |
| 9 | 6 (-1 + n) (85 - 296 n + 460 n^2 - 408 n^3 + 222 n^4 - 72 n^5 + 12 n^6) | |

| $m$ | Third differences of $m$-th powers: $D_{3(m)}$ | |
|---|---|---|
| 1 | | |
| 2 | | OFF. 3 |
| 3 | 6 | |
| 4 | 12 (-3 + 2 n) | |
| 5 | 30 (5 - 6 n + 2 n^2) | |
| 6 | 60 (-3 + 2 n) (3 - 3 n + n^2) | |
| 7 | 42 (43 - 90 n + 75 n^2 - 30 n^3 + 5 n^4) | |
| 8 | 84 (-3 + 2 n) (23 - 42 n + 32 n^2 - 12 n^3 + 2 n^4) | |
| 9 | 6 (3025 - 8694 n + 10836 n^2 - 7560 n^3 + 3150 n^4 - 756 n^5 + 84 n^6) | |

| $m$ | Fourth differences of $m$-th powers: $D_{4(m)}$ |
|---|---|
| 1 | |
| 2 | |
| 3 | |
| 4 | 24 |
| 5 | 120 (-2 + n) |
| 6 | 120 (13 - 12 n + 3 n^2) |
| 7 | 840 (-2 + n) (5 - 4 n + n^2) |
| 8 | 168 (243 - 400 n + 260 n^2 - 80 n^3 + 10 n^4) |
| 9 | 504 (-2 + n) (185 - 272 n + 164 n^2 - 48 n^3 + 6 n^4) |

OFF. 4

and we could continue indefinitely .

The so obtained polynomials, properly factored, provides formulas to calculate, at various orders, the differences of $m$-th powers.

Note that, from each table, you get sequences whose offset is equal to the differences order number.

## 0.6 Conclusion

In January 2015, I discovered that many of sequences obtained by the above procedures were not yet in the OEIS Encyclopedia database. This, in some way, gave me confirmation of newness of the introduced derivation procedures. I then immediately worked, contributing to the database OEIS with 53 new integer sequences, all approved and published in about two months. You can consult them here .