# ROW SUMS CONJECTURE IN ITERATED RASCAL TRIANGLES

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ABSTRACT. In this manuscript we review and prove the following conjecture for row sums of iterated rascal triangles. For every i

$$\sum_{k=0}^{4i+3} \binom{4i+3}{k}_i = 2^{4i+2}$$

where  $\binom{n}{k}_{i}$  is an iterated rascal number.

#### CONTENTS

1.	Introduction	1
2.	Row sums conjecture	3
3.	Acknowledgements	4
References		4

#### 1. INTRODUCTION

Rascal triangle is Pascal-like numeric triangle developed in 2010 by three middle school students, Alif Anggoro, Eddy Liu, and Angus Tulloch [1]. During math classes they were

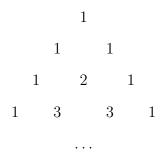
2010 Mathematics Subject Classification. 11B25,11B99.

*Key words and phrases.* Pascal's triangle, Rascal triangle, Binomial coefficients, Binomial identities, Binomial theorem, Generalized Rascal triangles, Iterated rascal triangles, Iterated rascal numbers, Number triangle, Arithmetic sequence, Vandermonde identity, Vandermonde convolution.

Date: July 5, 2024.

Sources: https://github.com/kolosovpetro/RowSumsConjectureInRascalTriangle

challenged to provide the next row for the following number triangle



Teacher's expected answer was the one that matches Pascal's triangle, e.g "1 4 6 4 1". However, Anggoro, Liu, and Tulloch suggested "1 4 5 4 1" instead. They devised this new row via what they called diamond formula

$$\mathbf{South} = \frac{\mathbf{East} \cdot \mathbf{West} + 1}{\mathbf{North}}$$

So they obtained the following triangle

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n/k	0	1	2	3	4	5	6	7
0	1							
1	1	1						
2	1	2	1					
3	1	3	3	1				
4	1	4	5	4	1			
5	1	5	7	7	5	1		
6	1	6	9	10	9	6	1	
$     \begin{array}{c}       0 \\       1 \\       2 \\       3 \\       4 \\       5 \\       6 \\       7     \end{array} $	1	7	11	13	13	11	7	1

Table 1. Rascal triangle.

Indeed, the forth row is "1 4 5 4 1" because  $4 = \frac{1\cdot3+1}{1}$  and  $5 = \frac{3\cdot3+1}{2}$ . Since then, a lot of work has been done over the topic of rascal triangles. In this article we stick our attention to the one of rascal triangles generalizations, namely iterated rascal triangles [2]. Iterated rascal number is defined via a sum of binomial coefficients multiplication

### **Definition 1.1.** Iterated rascal number

$$\binom{n}{k}_{i} = \sum_{m=0}^{i} \binom{n-k}{m} \binom{k}{m}$$
(1.1)

Thus, the rascal triangle (1) is the triangle generated by  $\binom{n}{k}_1$ .

### 2. Row sums conjecture

As we see iterated rascal triangles are indeed of Pascal-like triangles family. If rascal triangles are of Pascal-like triangles family, then similar properties must hold. I believe that is how the authors of [2] were thinking proposing the row sums conjecture for iterated rascal triangles.

Conjecture 2.1. (Conjecture 7.5 in [2].) For every i

$$\sum_{k=0}^{4i+3} \binom{4i+3}{k}_{i} = 2^{4i+2}$$

where  $\binom{n}{k}_i$  is an iterated rascal number.

*Proof.* Rewrite conjecture statement explicitly as

$$\sum_{k=0}^{4i+3} \sum_{m=0}^{i} \binom{4i+3-k}{m} \binom{k}{m} = 2^{4i+2}$$

Rearranging sums and omitting summation bounds yields

$$\sum_{m=0}^{i} \sum_{k} \binom{4i+3-k}{m} \binom{k}{m} = 2^{4i+2}$$
(2.1)

In Concrete mathematics [[3], p. 169, eq (5.26)], Knuth et al. provide the identity for the column sum of binomial coefficients multiplication

$$\sum_{k=0}^{l} \binom{l-k}{m} \binom{q+k}{n} = \binom{l+q+1}{m+n+1}$$
(2.2)

We can observe this pattern in the equation (2.1), thus the sum  $\sum_{k} \binom{4i+3-k}{m} \binom{k}{m}$  equals to

$$\sum_{k} \binom{4i+3-k}{m} \binom{k}{m} = \binom{4i+4}{2m+1}$$

Therefore, conjecture (2.1) is equivalent to

$$\sum_{m=0}^{i} \binom{4i+4}{2m+1} = 2^{4i+2}$$

Note that

$$\sum_{m=0}^{2i+1} \binom{4i+4}{2m+1} = 2^{4i+3}$$

So that

$$\frac{1}{2}\sum_{m=0}^{2i+1} \binom{4i+4}{2m+1} = \sum_{m=0}^{i} \binom{4i+4}{2m+1} = 2^{4i+2}$$

This completes the proof.

Therefore, the row sums conjecture for iterated rascal triangles is true for every row  $n = 4i + 3, i \ge 0.$ 

# **Proposition 2.2.** For every i

$$\sum_{k=0}^{4i+3} \binom{4i+3}{k}_i = 2^{4i+2}$$

### 3. Acknowledgements

Author is grateful to Oleksandr Kulkov, Markus Scheuer, Amelia Gibbs for their valuable feedback and suggestions regarding the row sums conjecture (2.1).

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# Version: 1.0.2-tags-v1-0-1.5+tags/v1.0.1.4020ca7

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