RELATION BETWEEN PASCAL'S TRIANGLE AND HYPERCUBES

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Abstract. In this short report famous binomial identity

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

is generalized and relation between Pascal's triangle and volume of m-dimension n-length Hypercube and number of its elements are shown. Where $(m,\ n)$ - positive integers.

Keywords. Binomial coefficient, Binomial Theorem, Pascal's triangle, Binomial Sum, Binomial distribution, Binomial Identity, Hypercube

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1. Introduction and main results

In this section let review and generalize well known fact about connection between row sums of Pascal triangle and 2−dimension Hypercube, recall property

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n \tag{1.1}$$

Now, let multiply each k-th term of of n-th row of Pascal's triangle [1] by 2^k

1				
1	2			
1	4	4		
1	6	12	8	
1	8	24	32	16

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Figure 5. Triangle built by $\binom{n}{k} \cdot 2^k, \ 0 \leq k \leq n \leq 4.$

We can notice that

$$\sum_{k=0}^{n} \binom{n}{k} \cdot 2^k = 3^n, \quad 0 \leq k \leq n, \quad (n, \ k) \in \mathbb{N} \tag{1.2}$$

Hereby, let be theorem

Theorem 1.3. *Volume of n-dimension hypercube with length m could be calculated as*

$$m^n = \sum_{k=0}^{n} \sum_{j=0}^{k} \binom{n}{k} \binom{k}{j} (-1)^{k-j} m^j \tag{1.4}$$

where m and n - positive integers, see [5].

Proof. Recall induction over m, in (1.1) is shown a well-known example for $m = 2$.

$$2^n = \sum_{k=0}^{n} \binom{n}{k} (2-1)^k \tag{1.5}$$

Review (1.5) and suppose that

$$(\underbrace{2+1}_{m=3})^n = \sum_{k=0}^{n} \binom{n}{k} \cdot (\underbrace{(2-1)+1}_{m-1})^k \tag{1.6}$$

And, obviously, this statement holds by means of Newton's Binomial Theorem [2], [3] given $m = 3$, more detailed, recall expansion for $(x+1)^n$ to show it.

$$(x+1)^n = \sum_{k=0}^{n} \binom{n}{k} x^k \tag{1.7}$$

Substituting $x = 2$ to (1.7) we have reached (1.6).
Next, let show example for each $m \in \mathbb{N}$. Recall Binomial theorem to show this

$$m^n = \sum_{k=0}^{n} \binom{n}{k} \cdot (m-1)^k \tag{1.8}$$

Hereby, for $m+1$ we receive Binomial theorem again

$$(m+1)^n = \sum_{k=0}^{n} \binom{n}{k} \cdot m^k \tag{1.9}$$

Review result from (1.8) and substituting Binomial expansion $\sum_{j=0}^{k} \binom{k}{j} (-1)^{n-k} m^j$ instead $(m-1)^k$ we receive desired result

$$\begin{aligned} m^n &= \sum_{k=0}^{n} \binom{n}{k} \cdot \underbrace{(m-1)^k}_{\sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} m^j} = \sum_{k=0}^{n} \binom{n}{k} \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} m^j \\ &= \sum_{k=0}^{n} \sum_{j=0}^{k} \binom{n}{k} \binom{k}{j} (-1)^{k-j} m^j \end{aligned} \tag{1.10}$$

This completes the proof. $\square$

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