PROOF OF KNUTH'S BINOMIAL IDENTITY (5.43)

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Abstract. In this manuscript we prove a remarkable Binomial identity (5.43) given by Donald Knuth in his fundamental work entitled Concrete Mathematics.

1. Introduction

In this manuscript we prove a remarkable Binomial identity (5.43) given by Donald Knuth in his fundamental work entitled Concrete Mathematics, see [1, p. 190]

Proposition 1.1 (Knuth's Binomial identity (5.43))**.** *For non-negative integers n, and for arbitrary integers s, r*

$$\sum_{k=0}^{n} \binom{n}{k} \binom{r-sk}{n} (-1)^k = s^n.$$

2. Proof of Knuth's Binomial identity

We begin our proof by recalling the formula for n-order finite differences of function f

$$\Delta^n f(x) = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} f(x+k).$$

Now, we define the binomial function F, such that

$$F(x) = \binom{r-sx}{n}.$$

Thus, the t-order forward finite difference of F is

$$\Delta^t F(x) = \sum_{k=0}^{t} \binom{t}{k} \binom{r-s(x+k)}{n} (-1)^{n-k}$$

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We may notice that the equation above quite reminds the structure of Knuth's binomial identity (1.1). By evaluating $\Delta^t F(x)$ in zero, we reach our goal even further, because

$$\Delta^t F(0) = \sum_{k=0}^{t} \binom{t}{k} \binom{r-sk}{n} (-1)^{n-k}.$$

Next, by setting $t = n$ gives the exact structure of (1.1), such that

$$\Delta^n F(0) = \sum_{k=0}^{n} \binom{n}{k} \binom{r-sk}{n} (-1)^{n-k}.$$

Still, it may not be immediately clear why $\Delta^n F(x) = (-1)^n s^n$. To make things work, we refer to the formula (5.42) in Concrete mathematics, see [1, p. 190],

$$\sum_k \binom{n}{k} (-1)^k \left[a_0 + a_1 k^1 + a_2 k^2 + \cdots + a_n k^n\right] = (-1)^n n! a_n,$$

which states that n-th difference of a polynomial of degree n in k equals to the coefficient of k^n multiplied by $(-1)^n n!$. This helps us a lot, because the binomial coefficient $\binom{r-sk}{n}$ is actually a polynomial of degree n in $r - sk$. The famous identity in Stirling numbers of the first kind $\left[{n \atop k}\right]$ shows it clearly

$$\begin{aligned}\binom{r-sk}{n} &= \frac{(-1)^0}{n!}\begin{bmatrix}n\\0\end{bmatrix}(r-sk)^0 + \frac{(-1)^1}{n!}\begin{bmatrix}n\\1\end{bmatrix}(r-sk)^1 + \cdots + \frac{(-1)^n}{n!}\begin{bmatrix}n\\n\end{bmatrix}(r-sk)^n \\ &= \sum_{j=0}^{n} \frac{(-1)^j}{n!}\begin{bmatrix}n\\j\end{bmatrix}(r-sk)^j.\end{aligned}$$

Thus, the coefficient a_n of k^n is

$$a_n = [k^n]\frac{(-1)^n}{n!}\begin{bmatrix}n\\n\end{bmatrix}(r-sk)^n = \frac{(-1)^n}{n!}\begin{bmatrix}n\\n\end{bmatrix}[k^n](r-sk)^n.$$

By Binomial theorem,

$$(r-sk)^n = \sum_{m=0}^{n} (-1)^{n-m}\binom{n}{m} r^m s^{n-m} k^{n-m}.$$

Thus,

$$[k^n](r-sk)^n = (-1)^{n-0}\binom{n}{0} r^0 s^{n-0}. = (-1)^n s^n$$

Hence, the coefficient a_n of k^n equals to

$$a_n = \frac{(-1)^n}{n!} \begin{bmatrix} n \\ n \end{bmatrix} (-1)^n s^n = \frac{s^n}{n!}.$$

Which implies that,

$$\Delta^n F(0) = \sum_{k=0}^{n} \binom{n}{k} \binom{r-sk}{n} (-1)^{n-k} = (-1)^n n! a_n = (-1)^n n! \frac{s^n}{n!}.$$

Therefore, the Knuth's identity (1.1) is indeed true

$$s^n = (-1)^n \Delta^n F(0) = \sum_{k=0}^{n} \binom{n}{k} \binom{r-sk}{n} (-1)^k.$$

This completes the proof of Proposition (1.1). □

It is quite remarkable that although the identity (1.1) is a special case of forward finite differences of $F(x) = \binom{r-sx}{n}$ evaluated in zero $s^n = (-1)^n \Delta^n F(0)$ – it holds for all x, because the coefficient of k^n remains s^n for all x

Proposition 2.1 (Generalized Knuths' binomial identity)**.** *For non-negative integers n, and for arbitrary integers s, r, x*

$$s^n = (-1)^n \Delta^n F(x) = \sum_{k=0}^{n} \binom{n}{k} \binom{r-sx-sk}{n} (-1)^k.$$

Because the coefficient of k^n in $\binom{r-sx-sk}{n}$ is

$$a_n = \frac{(-1)^n}{n!} \begin{bmatrix} n \\ n \end{bmatrix} [k^n]((r-sx)-sk)^n = \frac{s^n}{n!}.$$

Because, by Binomial theorem

$$[k^n]((r-sx)-sk)^n = (-1)^{n-0} \binom{n}{0} (r-sx)^0 s^{n-0} = (-1)^n s^n.$$

Conclusions

In this manuscript we have shown that the remarkable Binomial identity (1.1) given by Donald Knuth in his fundamental work entitled Concrete Mathematics is indeed true. In addition, we provide a generalization for identity (5.42) for all x, that is (2.1).

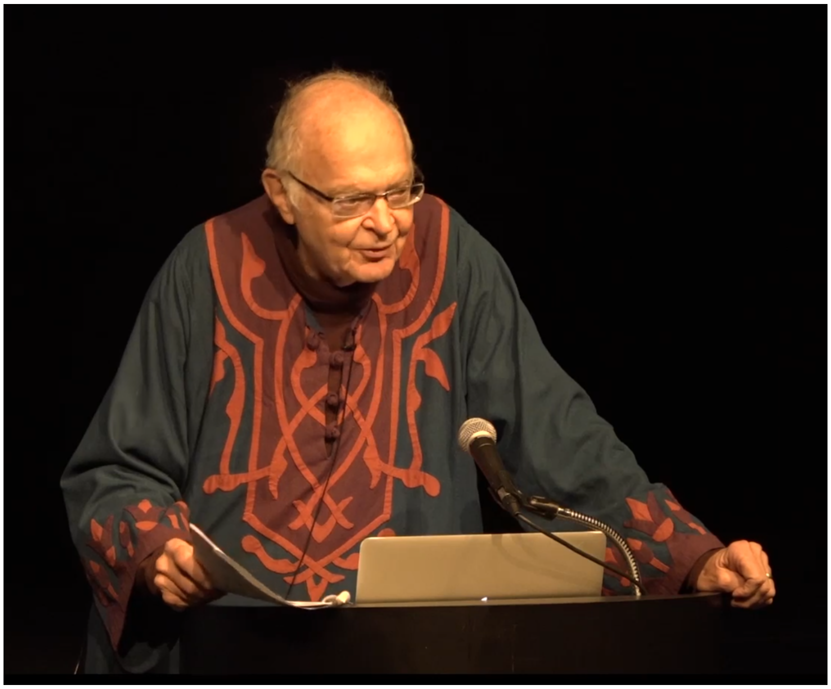

Figure 1. Professor Knuth at Computer Science, the Bible, and Music - 2018 Lectures.

References

[1] Graham, Ronald L. and Knuth, Donald E. and Patashnik, Oren. *Concrete mathematics: A foundation for computer science (second edition)*. Addison-Wesley Publishing Company, Inc., 1994. https://archive.org/details/concrete-mathematics.

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