EXTENDED TABLES OF $U_m(n,k)$ COEFFICIENTS

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ABSTRACT. In this short report we briefly describe the coefficient $U_m(n,k)$ for $m=1,2,3,4$ and attach extended tables, containing mentioned the polynomials consisting already mentioned coefficients $U_m(n,k),\ m=1,2,3,4$.

1. Introduction and Main results

Review the main result of [1], that is the identity

$$n^{2m+1} = \sum_{1\le k\le n}\sum_{j\ge 0} A_{m,j} k^j (n-k)^j, \tag{1.1}$$

where $A_{m,j}$ is from sequences A302971 and A304042. In this short report we examine the polynomial

$$\sum_{0\le k\le m} (-1)^{m-k} U_m(n,k)\cdot n^k, \tag{1.2}$$

That is generated by the

$$\sum_{1\le k\le T}\sum_{j\ge 0} A_{m,j} k^j (n-k)^j = \sum_{0\le k\le m} (-1)^{m-k} U_m(n,k)\cdot n^k, \tag{1.3}$$

where $T=1,2,3...$ and $m\ge 0,\ m=\text{const}$. The coefficient $A_{m,j}$ is generated by

$$A_{m,j} := \begin{cases} 0, & \text{if } j<0 \text{ or } j>m \\ (2j+1)\binom{2j}{j}\sum_{d=2j+1}^{m} A_{m,d}\binom{d}{2j+1}\frac{(-1)^{d-1}}{d-j}B_{2d-2j}, & \text{if } 0\le j<m \\ (2j+1)\binom{2j}{j}, & \text{if } j=m \end{cases}$$

Derivation of coefficient $A_{m,j}$ is discussed in [2] and [1]. In particular, the right part of (1.3) returns odd power $2m+1$ of $T\in\mathbb{N}$ when $n=T$

$$T^{2m+1} = \sum_{0\le k\le m} (-1)^{m-k} U_m(T,k)\cdot T^k$$

1.1. **Detailed derivation of the polynomials, consisting the coefficient $U_m(n,k)$.** Consider the identity discussed in [2],

$$n^{2m+1} = \sum_{1\le k\le n}\sum_{j\ge 0} A_{m,j} k^j (n-k)^j, \tag{1.4}$$

Let show a few examples of polynomials $\sum_{j\ge 0} A_{m,j} k^j (n-k)^j$ for $m=1,2,3$. We denote the part $\sum_{j\ge 0} A_{m,j} k^j (n-k)^j$ of the left part of equation (1.3) as

$$D_m(n,k) = \sum_{j\ge 0} A_{m,j} k^j (n-k)^j \tag{1.5}$$

Therefore,

$$\begin{cases} D_1(n,k) = 1+6k(n-k), & \text{for } A287326 \\ D_2(n,k) = 1-0k(n-k)+30k^2(n-k)^2, & \text{for } A300656 \\ D_3(n,k) = 1-14k(n-k)+0k^2(n-k)^2+140k^3(n-k)^3, & \text{for } A300785 \end{cases} \tag{1.6}$$

The coefficients in $D_{t=1,2,3}(n,k)$ are the terms of corresponding row of triangle A302971. Now, we show an example of generation of polynomials from the right part of (1.3) for $m=1$,

Example 1.7. Let be $m=1$, then we rewrite the left hand side of (1.3) as

$$\sum_{1\le k\le T}\sum_{j\ge 0} A_{1,j} k^j (n-k)^j \tag{1.8}$$

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Next, let substitute the polynomial $D_1(n,k)$ from (1.6) into left hand side of equation (1.8) and let be $T=1,...,10$, then

$$\sum_{1\leq k\leq T} 1+6k(n-k) = \begin{cases} T=1: & -5+6n \\ T=2: & -28+18n \\ T=3: & -81+36n \\ T=4: & -176+60n \\ T=5: & -325+90n \\ T=6: & -540+126n \\ T=7: & -833+168n \\ T=8: & -1216+216n \\ T=9: & -1701+270n \\ T=10: & -2300+330n \end{cases} \tag{1.9}$$

Let show the case for $m=2$ and $T=1,...,10$, again we recall the corresponding polynomial $D_2(n,k)$ from (1.6) and substitute it into left part of (1.3),

$$\sum_{1\leq k\leq T} 1-0k(n-k)+30k^2(n-k)^2 = \begin{cases} T=1: & 31-60n+30n^2 \\ T=2: & 512-540n+150n^2 \\ T=3: & 2943-2160n+420n^2 \\ T=4: & 10624-6000n+900n^2 \\ T=5: & 29375-13500n+1650n^2 \\ T=6: & 68256-26460n+2730n^2 \\ T=7: & 140287-47040n+4200n^2 \\ T=8: & 263168-77760n+6120n^2 \\ T=9: & 459999-121500n+8550n^2 \\ T=10: & 760000-181500n+11550n^2 \end{cases} \tag{1.10}$$

Similarly, let show an example for $m=3$ and $T=1,...,10$,

$$\sum_{1\leq k\leq T} 1-14k(n-k)+0k^2(n-k)^2+140k^3(n-k)^3 = \begin{cases} T=1: & -125+406n-420n^2+140n^3 \\ T=2: & -9028+13818n-7140n^2+1260n^3 \\ T=3: & -110961+115836n-41160n^2+5040n^3 \\ T=4: & -684176+545860n-148680n^2+14000n^3 \\ T=5: & -2871325+1858290n-411180n^2+31500n^3 \\ T=6: & -9402660+5124126n-955500n^2+61740n^3 \\ T=7: & -25872833+12182968n-1963920n^2+109760n^3 \\ T=8: & -62572096+25945416n-3684240n^2+181440n^3 \\ T=9: & -136972701+50745870n-6439860n^2+283500n^3 \\ T=10: & -276971300+92745730n-10639860n^2+423500n^3 \end{cases} \tag{1.11}$$

We can observe, that in every upper example, the resulting polynomial for every m,T has the following form

$$\sum_{0\leq k\leq m} (-1)^{m-k} U_m(n,k)\cdot n^k, \tag{1.12}$$

therefore, the following question is stated

Question 1.13. *Is there a recurrent that gives the coefficients $U_m(n,k)$ otherwise then by the identity*

$$\sum_{1\leq k\leq T}\sum_{j\geq 0} A_{m,j}k^j(n-k)^j = \sum_{0\leq k\leq m} (-1)^{m-k} U_m(n,k)\cdot n^k, \tag{1.14}$$

i.e is there any function $F(n,m)$ such that $F(m,n)=U_m(n,k)$?

2. Extended tables of polynomials, containing the coefficient $U_m(n,k)$, for $m=1,2,3,4$.

Therefore, below we attach a tables containing the polynomials

$$\sum_{1\leq k\leq T}\sum_{j\geq 0} A_{m,j}k^j(n-k)^j = \sum_{0\leq k\leq m} (-1)^{m-k} U_m(n,k)\cdot n^k$$

for various T and m, that is shown in each table's caption. The following tables could be generated using Mathematica code Um(n,k)_coefficients2.txt. Here we begin to show our tables for $m=1,2,3,4$

T	Polynomial $\sum_{0\leq k\leq 1} (-1)^{1-k} U_1(n,k)\cdot n^k$

1	$-5+6n$
2	$-28+18n$
3	$-81+36n$
4	$-176+60n$
5	$-325+90n$
6	$-540+126n$
7	$-833+168n$
8	$-1216+216n$
9	$-1701+270n$
10	$-2300+330n$
11	$-3025+396n$
12	$-3888+468n$
13	$-4901+546n$
14	$-6076+630n$
15	$-7425+720n$
16	$-8960+816n$
17	$-10693+918n$
18	$-12636+1026n$
19	$-14801+1140n$
20	$-17200+1260n$
21	$-19845+1386n$
22	$-22748+1518n$
23	$-25921+1656n$
24	$-29376+1800n$
25	$-33125+1950n$
26	$-37180+2106n$
27	$-41553+2268n$
28	$-46256+2436n$
29	$-51301+2610n$
30	$-56700+2790n$

Table 1: **Case 1.** Table for $m=1$, generating function: $\sum_{1\le k\le T} D_m(n,k)$ over $T=1,2,...,30$

T	Polynomial $\sum\limits_{0\le k\le 2}(-1)^{2-k}U_2(n,k)\cdot n^k$
1	$31-60n+30n^2$
2	$512-540n+150n^2$
3	$2943-2160n+420n^2$
4	$10624-6000n+900n^2$
5	$29375-13500n+1650n^2$
6	$68256-26460n+2730n^2$
7	$140287-47040n+4200n^2$
8	$263168-77760n+6120n^2$
9	$459999-121500n+8550n^2$
10	$760000-181500n+11550n^2$
11	$1199231-261360n+15180n^2$
12	$1821312-365040n+19500n^2$
13	$2678143-496860n+24570n^2$
14	$3830624-661500n+30450n^2$
15	$5349375-864000n+37200n^2$
16	$7315456-1109760n+44880n^2$
17	$9821087-1404540n+53550n^2$
18	$12970368-1754460n+63270n^2$
19	$16879999-2166000n+74100n^2$
20	$21680000-2646000n+86100n^2$
21	$27514431-3201660n+99330n^2$
22	$34542112-3840540n+113850n^2$
23	$42937343-4570560n+129720n^2$
24	$52890624-5400000n+147000n^2$
25	$64609375-6337500n+165750n^2$
26	$78318656-7392060n+186030n^2$
27	$94261887-8573040n+207900n^2$

28	$112701568 - 9890160n + 231420n^2$
29	$133919999 - 11353500n + 256650n^2$
30	$158220000 - 12973500n + 283650n^2$

Table 2: **Case 2.** Table for $m = 2$, generating function: $\sum_{1 \leq k \leq T} D_m(n, k)$ over $T = 1, 2, ..., 30$

T	Polynomial $\sum\limits_{0 \leq k \leq 3} (-1)^{3-k} U_3(n, k) \cdot n^k$
1	$-125 + 406n - 420n^2 + 140n^3$
2	$-9028 + 13818n - 7140n^2 + 1260n^3$
3	$-110961 + 115836n - 41160n^2 + 5040n^3$
4	$-684176 + 545860n - 148680n^2 + 14000n^3$
5	$-2871325 + 1858290n - 411180n^2 + 31500n^3$
6	$-9402660 + 5124126n - 955500n^2 + 61740n^3$
7	$-25872833 + 12182968n - 1963920n^2 + 109760n^3$
8	$-62572096 + 25945416n - 3684240n^2 + 181440n^3$
9	$-136972701 + 50745870n - 6439860n^2 + 283500n^3$
10	$-276971300 + 92745730n - 10639860n^2 + 423500n^3$
11	$-524988145 + 160386996n - 16789080n^2 + 609840n^3$
12	$-943023888 + 264896268n - 25498200n^2 + 851760n^3$
13	$-1618774781 + 420839146n - 37493820n^2 + 1159340n^3$
14	$-2672907076 + 646725030n - 53628540n^2 + 1543500n^3$
15	$-4267591425 + 965662320n - 74891040n^2 + 2016000n^3$
16	$-6616398080 + 1406064016n - 102416160n^2 + 2589440n^3$
17	$-9995653693 + 2002403718n - 137494980n^2 + 3277260n^3$
18	$-14757360516 + 2796022026n - 181584900n^2 + 4093740n^3$
19	$-21343778801 + 3835983340n - 236319720n^2 + 5054000n^3$
20	$-30303773200 + 5179983060n - 303519720n^2 + 6174000n^3$
21	$-42311023965 + 6895305186n - 385201740n^2 + 7470540n^3$
22	$-58184203748 + 9059830318n - 483589260n^2 + 8961260n^3$
23	$-78909220801 + 11763094056n - 601122480n^2 + 10664640n^3$
24	$-105663629376 + 15107395800n - 740468400n^2 + 12600000n^3$
25	$-139843308125 + 19208957950n - 904530900n^2 + 14787500n^3$
26	$-183091507300 + 24199135506n - 1096460820n^2 + 17248140n^3$
27	$-237330365553 + 30225676068n - 1319666040n^2 + 20003760n^3$
28	$-304794997136 + 37454030236n - 1577821560n^2 + 23077040n^3$
29	$-388070250301 + 46068712410n - 1874879580n^2 + 26491500n^3$
30	$-490130237700 + 56274711990n - 2215079580n^2 + 30271500n^3$
31	$-614380739585 + 68298954976n - 2602958400n^2 + 34442240n^3$
32	$-764704580608 + 82391815968n - 3043360320n^2 + 39029760n^3$
33	$-945510081021 + 98828680566n - 3541447140n^2 + 44060940n^3$
34	$-1161782683076 + 117911558170n - 4102708260n^2 + 49563500n^3$
35	$-1419139853425 + 139970745180n - 4732970760n^2 + 55566000n^3$
36	$-1723889362320 + 165366538596n - 5438409480n^2 + 62097840n^3$
37	$-2083091040413 + 194491000018n - 6225557100n^2 + 69189260n^3$
38	$-2504622113956 + 227769770046n - 7101314220n^2 + 76871340n^3$
39	$-2997246219201 + 265663933080n - 8072959440n^2 + 85176000n^3$
40	$-3570686196800 + 308671932520n - 9148159440n^2 + 94136000n^3$

Table 3: **Case 3.** For $m = 3$, generating function: $\sum_{1 \leq k \leq T} D_m(n, k)$ over $T = 1, 2, ..., 40$

T	Polynomial $\sum\limits_{0 \leq k \leq 4} (-1)^{4-k} U_4(n, k) \cdot n^k$
1	$751 - 2640n + 3780n^2 - 2520n^3 + 630n^4$
2	$162512 - 325440n + 245700n^2 - 83160n^3 + 10710n^4$
3	$4297023 - 5837040n + 3001320n^2 - 695520n^3 + 61740n^4$
4	$45586624 - 47125200n + 18484200n^2 - 3276000n^3 + 223020n^4$
5	$291683375 - 244000800n + 77546700n^2 - 11151000n^3 + 616770n^4$
6	$1349845776 - 949440240n + 253906380n^2 - 30746520n^3 + 1433250n^4$
7	$4981676287 - 3024769440n + 698619600n^2 - 73100160n^3 + 2945880n^4$
8	$15551330048 - 8309593440n + 1689523920n^2 - 155675520n^3 + 5526360n^4$

9	$42670773999 - 20362676400n + 3698370900n^2 - 304479000n^3 + 9659790n^4$
10	$105670786000 - 45562677600n + 7478370900n^2 - 556479000n^3 + 15959790n^4$
11	$240716895551 - 94670349840n + 14174871480n^2 - 962327520n^3 + 25183620n^4$
12	$511605381312 - 184966507440n + 25461891000n^2 - 1589384160n^3 + 38247300n^4$
13	$1025515755823 - 343092771840n + 43707229020n^2 - 2525042520n^3 + 56240730n^4$
14	$1955262884624 - 608734803600n + 72168875100n^2 - 3880359000n^3 + 80442810n^4$
15	$3569884005375 - 1039300430400n + 115225437600n^2 - 5793984000n^3 + 112336560n^4$
16	$6275713432576 - 1715757781440n + 178643314080n^2 - 8436395520n^3 + 153624240n^4$
17	$10670440655087 - 2749811239440n + 269883324900n^2 - 12014435160n^3 + 206242470n^4$
18	$17613015856848 - 4292605722240n + 398449531620n^2 - 16776146520n^3 + 272377350n^4$
19	$28312660615999 - 6545162506800n + 576282961800n^2 - 23015916000n^3 + 354479580n^4$
20	$44440660664000 - 9770762509200n + 818202961800n^2 - 31079916000n^3 + 455279580n^4$
21	$68269062114351 - 14309505635040n + 1142398899180n^2 - 41371850520n^3 + 577802610n^4$
22	$102840862500112 - 20595287515440n + 1570974936300n^2 - 54359003160n^3 + 725383890n^4$
23	$152176783290623 - 29175447644640n + 2130550596720n^2 - 70578587520n^3 + 901683720n^4$
24	$221524231290624 - 40733355636000n + 2852919846000n^2 - 90644400000n^3 + 1110702600n^4$
25	$317654602459375 - 56114215014000n + 3775771408500n^2 - 115253775000n^3 + 1356796350n^4$
26	$449215653223376 - 76354376660640n + 4943473041780n^2 - 145194842520n^3 + 1644691230n^4$
27	$627146261293887 - 102714466735440n + 6407922490200n^2 - 181354088160n^3 + 1979499060n^4$
28	$865161520339648 - 136716646589040n + 8229467839320n^2 - 224724215520n^3 + 2366732340n^4$
29	$1180316760605999 - 180186334891200n + 10477899992700n^2 - 276412311000n^3 + 2812319370n^4$
30	$1593659760714000 - 235298734894800n + 13233519992700n^2 - 337648311000n^3 + 3322619370n^4$
31	$2130981114417151 - 304630522458240n + 16588283906880n^2 - 409793771520n^3 + 3904437600n^4$
32	$2823673440038912 - 391217063149440n + 20647028001600n^2 - 494350940160n^3 + 4565040480n^4$
33	$3709710869661423 - 498615539455440n + 25528776924420n^2 - 592972130520n^3 + 5312170710n^4$
34	$4834761029884624 - 630974381822400n + 31368137616900n^2 - 707469399000n^3 + 6154062390n^4$
35	$6253442526125375 - 793109409951600n + 38316781679400n^2 - 839824524000n^3 + 7099456140n^4$
36	$8030741767978176 - 990587103477840n + 46545018909480n^2 - 992199287520n^3 + 8157614220n^4$
37	$10243603824112687 - 1229815433857440n + 56243464735500n^2 - 1166946059160n^3 + 9338335650n^4$
38	$12982712871538448 - 1518142701993840n + 67624804267020n^2 - 1366618682520n^3 + 10651971330n^4$
39	$16354478705823999 - 1863964838829600n + 80925655683600n^2 - 1593983664000n^3 + 12109439160n^4$
40	$20483246706016000 - 2276841638834400n + 96408535683600n^2 - 1852031664000n^3 + 13722239160n^4$

Table 4: **Case 4.** For $m = 4$, generating function: $\sum_{1 \le k \le T} D_m(n,k)$ over $T = 1, 2, ..., 40$

References

[1] Petro Kolosov. Series Representation of Power Function., 2018. arXiv preprint: arXiv:1603.02468 [math.NT].

[2] Discussion on $A_{m,j}$ coeffcients. https://mathoverflow.net/questions/297900/