ON THE SUMMATION OF DIVERGENT SERIES

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Abstract. Divergent series has always been an exciting topic. Sometimes the summation of divergent series might seem confusing and surprising. In this short report, we provide an overview of the most popular ways to calculate the sum of divergent series among with concrete examples of such sums.

Contents

1. Introduction

In this manuscript we review divergent series summation from heuristic prospective, furthermore, we discuss summation of divergent series by means of Ramanujan summation method, and, finally, we review a zeta function in terms of divergent series.

2. Heuristic background

In chapter 8 of the first collection of his writings, Ramanujan showed that $1+2+3+4+\cdots = \frac{-1}{12}$ using two methods. The first key observation is that the series $1+2+3+4+\cdots$ is

Date: July 18, 2023.

2010 *Mathematics Subject Classification.* 26E70, 05A30.

Key words and phrases. Ramanujan summation, divergent series .

similar to the alternating series of natural numbers $1 - 2 + 3 - 4 + \cdots$. Although this series is also divergent, it is much easier to work with. There are several classical ways to assign a final value to this series, known since the 18th century.

In order to bring the series $1 + 2 + 3 + 4 + \cdots$ to the form $1 - 2 + 3 - 4 + \cdots$, we can subtract 4 from the second term, 8 from the fourth term, 12 from the sixth, etc. The total value to be subtracted is expressed by the series $4 + 8 + 12 + 16 + \cdots$, which is obtained by multiplying the original series $1 + 2 + 3 + 4 + \cdots$ by 4. Let be c

$$c = 1 + 2 + 3 + 4 + \cdots$$

Then $4c$ is

$$4c = 4 + 8 + 12 + 16 + \cdots$$

If we subtract $4c$ from c we get

$$-3c = 1 - 2 + 3 - 4 + \cdots \tag{2.1}$$

Here we notice that $1 - 2 + 3 - 4 + \cdots$ is Taylor series $T(x)$ of $f(x) = \frac{1}{(1+x)^2}$ for $x = 1$, eg

$$T(x) = 1 - 2x + 3x^2 - 4x^3 + 5x^4 - 6x^5 + O(x^6)$$

$$T(1) = 1 - 2 + 3 - 4 + 5 - 6 + \cdots$$

Therefore, equation (2.1) turns to

$$-3c = 1 - 2 + 3 - 4 + \cdots = \frac{1}{(1+1)^2} = \frac{1}{4}$$

Finally, the sum of natural series is

$$c = 1 + 2 + 3 + 4 + \cdots = -\frac{1}{12}.$$

Furthermore, this approach was extended to Ramanujan's summation method, which involves Euler-Maclaurin formula.

3. Ramanujan's summation formula

Ramanujan summation essentially is a property of the partial sums, rather than a property of the entire sum, as that doesn't exist. If we take the Euler-Maclaurin summation formula together with the correction rule using Bernoulli numbers, we see that

$$\begin{aligned}\frac{1}{2}f(0) + f(1) + \cdots + f(n-1) + \frac{1}{2}f(n) &= \frac{1}{2}[f(0) + f(n)] + \sum_{k=1}^{n-1} f(k) \\ &= \int_0^n f(x)\,dx + \sum_{k=1}^{p} \frac{B_{k+1}}{(k+1)!}\left[f^{(k)}(n) - f^{(k)}(0)\right] + R_p\end{aligned}$$

Ramanujan wrote it for the case p going to infinity

$$\sum_{k=1}^{x} f(k) = C + \int_0^x f(t)\,dt + \frac{1}{2}f(x) + \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!} f^{(2k-1)}(x)$$

Therefore, yet again we get a value of natural series equals to $-\frac{1}{12}$

$$\sum_{k=1}^{\infty} k = -\frac{1}{2}f(0) - \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!} f^{(2k-1)}(0) = \frac{1}{6}\cdot\frac{1}{2!} = -\frac{1}{12}$$

4. Riemann zeta function

We continue our journey to the world of divergent series from the one very widely-known formula, it's called a Riemann zeta function $\zeta(s)$. For every $s \in \mathbb{R}$ Riemann zeta function is defined as

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

The following identity holds in terms of zeta function

$$\zeta(1-N) = -\frac{B_N}{N}$$

Consider the case $N = 1$

$$\sum_{n=1}^{\infty} 1 = \zeta(1-N) = -\frac{B_1}{1} = \frac{1}{2}$$

since Bernoulli number $B_1 = \frac{1}{2}$. Similarly for $N = 2, 3$

$$\sum_{n=1}^{\infty} n = \zeta(1-2) = -\frac{B_2}{2} = -\frac{1}{12}$$

$$\sum_{n=1}^{\infty} n^2 = \zeta(1-3) = -\frac{B_3}{3} = 0$$

$$\sum_{n=1}^{\infty} n^3 = \zeta(1-4) = -\frac{B_4}{4} = \frac{1}{120}$$

In general,

$$\sum_{k=1}^{\infty} k^N = \zeta(1-N) = -\frac{B_N}{N} = \text{const}$$

5. Conclusions

In this report, we have discussed the ways to sum divergent series.

Version: `0.1.2-tags-v0-1-0.0+Branch.tags-v0.1.0.Sha.210424a3e1469e6a72782a6ddd3c760f1b97`

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