DISCUSSION ON COEFFICIENTS OF ODD POLYNOMIAL IDENTITY

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ABSTRACT. https://mathoverflow.net/a/297916/113033

1. Introduction

Assuming that following odd power identity holds

$$n^{2m+1} = \sum_{r=0}^{m} \sum_{k=1}^{n} \mathbf{A}_{m,r} k^r (n-k)^r \tag{1}$$

Our main goal is to identify the set of coefficients $\mathbf{A}_{m,0}, \mathbf{A}_{m,1}, \dots, \mathbf{A}_{m,m}$ such that identity above is true.

Although, the recurrence relation is already given at [1], a few key points in proof are worth to discuss additionally.

The main idea of Alekseyev's approach was to utilize dynamic programming methods to evaluate the $\mathbf{A}_{m,r}$ recursively, taking the base case $\mathbf{A}_{m,m}$ and then evaluating the next coefficient $\mathbf{A}_{m,m-1}$ by using backtracking, continuing similarly up to $\mathbf{A}_{m,0}$.

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By applying Binomial theorem $(n-k)^r = \sum_{t=0}^{r}(-1)^t\binom{r}{t}n^{r-t}k^t$ and Faulhaber's formula $\sum_{k=1}^{n} k^p = \left[\frac{1}{p+1}\sum_j \binom{p+1}{j} B_j n^{p+1-j}\right] - B_{p+1}$, we get

$$\begin{aligned}
\sum_{k=1}^{n} k^r(n-k)^r &= \sum_{t=0}^{r}(-1)^t\binom{r}{t}n^{r-t}\sum_{k=1}^{n}k^{t+r} \\
&= \sum_{t=0}^{r}(-1)^t\binom{r}{t}n^{r-t}\left[\frac{1}{t+r+1}\sum_j\binom{t+r+1}{j}B_j n^{t+r+1-j} - B_{t+r+1}\right] \\
&= \sum_{t=0}^{r}\binom{r}{t}\left[\frac{(-1)^t}{t+r+1}\sum_j\binom{t+r+1}{j}B_j n^{2r+1-j} - B_{t+r+1}n^{r-t}\right] \\
&= \left[\sum_{t=0}^{r}\binom{r}{t}\frac{(-1)^t}{t+r+1}\sum_j\binom{t+r+1}{j}B_j n^{2r+1-j}\right] - \left[\sum_{t=0}^{r}\binom{r}{t}\frac{(-1)^t}{t+r+1}B_{t+r+1}n^{r-t}\right] \\
&= \left[\sum_{j,t}\binom{r}{t}\frac{(-1)^t}{t+r+1}\binom{t+r+1}{j}B_j n^{2r+1-j}\right] - \left[\sum_{t}\binom{r}{t}\frac{(-1)^t}{t+r+1}B_{t+r+1}n^{r-t}\right]
\end{aligned}$$

Rearranging terms yields

$$\left[\sum_j B_j n^{2r+1-j}\sum_t\binom{r}{t}\frac{(-1)^t}{t+r+1}\binom{t+r+1}{j}\right] - \left[\sum_t\binom{r}{t}\frac{(-1)^t}{t+r+1}B_{t+r+1}n^{r-t}\right] \tag{2}$$

We can notice that

$$\sum_t\binom{r}{t}\frac{(-1)^t}{r+t+1}\binom{r+t+1}{j} = \begin{cases}\frac{1}{(2r+1)\binom{2r}{r}} & \text{if } j=0\\ \frac{(-1)^r}{j}\binom{r}{2r-j+1} & \text{if } j>0\end{cases} \tag{3}$$

An elegant proof of the binomial identity (3) is presented in [2].

In particular, equation (3) is zero for $0 < t \le j$. By using (3), we are able to move $j=0$ out of summation in (2) which yields

$$\begin{aligned}
\sum_{k=1}^{n} k^r(n-k)^r = \frac{1}{(2r+1)\binom{2r}{r}}n^{2r+1} + \left[\sum_{j\ge 1}B_j n^{2r+1-j}\sum_t\binom{r}{t}\frac{(-1)^t}{t+r+1}\binom{t+r+1}{j}\right] \\
- \left[\sum_{t=0}^{r}\binom{r}{t}\frac{(-1)^t}{t+r+1}B_{t+r+1}n^{r-t}\right]
\end{aligned}$$

Simplifying above equation by using (3) yields

$$\sum_{k=1}^{n} k^r (n-k)^r = \frac{1}{(2r+1)\binom{2r}{r}} n^{2r+1} + \underbrace{\left[\sum_{j\geq 1} \frac{(-1)^r}{j} \binom{r}{2r-j+1} B_j n^{2r-j+1}\right]}_{(\star)} - \underbrace{\left[\sum_{t=0}^{r} \binom{r}{t} \frac{(-1)^t}{t+r+1} B_{t+r+1} n^{r-t}\right]}_{(\diamond)}$$

Hence, introducing $\ell = 2r - j + 1$ to $(\star)$ and $\ell = r - t$ to $(\diamond)$ we collapse the common terms

$$\begin{aligned}\sum_{k=1}^{n} k^r (n-k)^r &= \frac{1}{(2r+1)\binom{2r}{r}} n^{2r+1} + \left[\sum_{\ell} \frac{(-1)^r}{2r+1-\ell} \binom{r}{\ell} B_{2r+1-\ell} n^{\ell}\right] \\ &\quad - \left[\sum_{\ell} \binom{r}{\ell} \frac{(-1)^{r-\ell}}{2r+1-\ell} B_{2r+1-\ell} n^{\ell}\right] \\ &= \frac{1}{(2r+1)\binom{2r}{r}} n^{2r+1} + 2 \sum_{\text{odd } \ell} \frac{(-1)^r}{2r+1-\ell} \binom{r}{\ell} B_{2r+1-\ell} n^{\ell}\end{aligned}$$

Assuming that $\mathbf{A}_{m,r}$ is defined by (1), we obtain the following relation for polynomials in n

$$\sum_{r} \mathbf{A}_{m,r} \frac{1}{(2r+1)\binom{2r}{r}} n^{2r+1} + 2 \sum_{r,\text{ odd } \ell} \mathbf{A}_{m,r} \frac{(-1)^r}{2r+1-\ell} \binom{r}{\ell} B_{2r+1-\ell} n^{\ell} \equiv n^{2m+1}$$

Replacing odd ℓ by k we get

$$\sum_{r} \mathbf{A}_{m,r} \frac{1}{(2r+1)\binom{2r}{r}} n^{2r+1} + 2 \sum_{r,\, k} \mathbf{A}_{m,r} \frac{(-1)^r}{2r-2k} \binom{r}{2k+1} B_{2r-2k} n^{2k+1} \equiv n^{2m+1} \tag{4}$$

Taking the coefficient of n^{2m+1} we get

$$\mathbf{A}_{m,m} = (2m+1)\binom{2m}{m} \tag{5}$$

Taking the coefficient of n^{2d+1} for an integer d in the range $\frac{m}{2} \leq d < m$, we get

$$\mathbf{A}_{m,d} = 0 \tag{6}$$

Taking the coefficient of n^{2d+1} for d in the range $\frac{m}{4} \leq d < \frac{m}{2}$ we get

$$\mathbf{A}_{m,d} \frac{1}{(2d+1)\binom{2d}{d}} + 2(2m+1)\binom{2m}{m}\binom{m}{2d+1} \frac{(-1)^m}{2m-2d} B_{2m-2d} = 0 \tag{7}$$

i.e

$$\mathbf{A}_{m,d} = (-1)^{m-1} \frac{(2m+1)!}{d!d!m!(m-2d-1)!} \frac{1}{m-d} B_{2m-2d}$$

Continue similarly we can compute $\mathbf{A}_{m,r}$ for each integer r in range $\frac{m}{2^{s+1}} \leq r < \frac{m}{2^s}$, iterating consecutively over $s = 1, 2, \dots$ by using previously determined values of $\mathbf{A}_{m,d}$ as follows

$$\mathbf{A}_{m,r} = (2r+1)\binom{2r}{r} \sum_{d \geq 2r+1}^{m} \mathbf{A}_{m,d} \binom{d}{2r+1} \frac{(-1)^{d-1}}{d-r} B_{2d-2r}$$

Finally, we are capable to define the following recurrence relation for coefficient $\mathbf{A}_{m,r}$

Definition 1.1. *(Definition of coefficient $\mathbf{A}_{m,r}$.)*

$$\mathbf{A}_{m,r} = \begin{cases} (2r+1)\binom{2r}{r} & \text{if } r = m \\ (2r+1)\binom{2r}{r} \sum_{d \geq 2r+1}^{m} \mathbf{A}_{m,d} \binom{d}{2r+1} \frac{(-1)^{d-1}}{d-r} B_{2d-2r} & \text{if } 0 \leq r < m \\ 0 & \text{if } r < 0 \text{ or } r > m \end{cases} \tag{8}$$

where B_t are Bernoulli numbers [3]. It is assumed that $B_1 = \frac{1}{2}$.

For example,

m/r	0	1	2	3	4	5	6	7
0	1							
1	1	6						
2	1	0	30					
3	1	-14	0	140				
4	1	-120	0	0	630			
5	1	-1386	660	0	0	2772		
6	1	-21840	18018	0	0	0	12012	
7	1	-450054	491400	-60060	0	0	0	51480

Table 1. Coefficients $\mathbf{A}_{m,r}$. See OEIS sequences [4, 5].

Properties of the coefficients $\mathbf{A}_{m,r}$

- $\mathbf{A}_{m,m} = \binom{2m}{m}$
- $\mathbf{A}_{m,r} = 0$ for $m < 0$ and $r > m$

- $\mathbf{A}_{m,r} = 0$ for $r < 0$
- $\mathbf{A}_{m,r} = 0$ for $\frac{m}{2} \leq r < m$
- $\mathbf{A}_{m,0} = 1$ for $m \geq 0$
- $\mathbf{A}_{m,r}$ are integers for $m \leq 11$
- Row sums: $\sum_{r=0}^{m} \mathbf{A}_{m,r} = 2^{2m+1} - 1$

2. Questions

Question 2.1. *Although, a proof of combinatorial identity (3) is already present, it is good to point out literature or more context on it. Reference to a book or article with deeper discussion.*

Question 2.2. *I have struggle to understand the equation (5), it takes the coefficient of n^{2m+1} meaning that we substitute $r = m$ into (4) evaluating it, if I understand it properly. So that coefficient of n^{2m+1} is*

$$\mathbf{A}_{m,m} \frac{1}{(2m+1)\binom{2m}{m}} n^{2m+1} + 2 \sum_{k} \mathbf{A}_{m,m} \frac{(-1)^m}{2m-2k} \binom{m}{2k+1} B_{2m-2k} n^{2k+1} = 1$$

It implies that coefficient of n^{2m+1} is zero

$$2 \sum_{k} \mathbf{A}_{m,m} \frac{(-1)^m}{2m-2k} \binom{m}{2k+1} B_{2m-2k} n^{2k+1} = 0$$

So that

$$\mathbf{A}_{m,m} \frac{1}{(2m+1)\binom{2m}{m}} = 1; \qquad \mathbf{A}_{m,m} = (2m+1)\binom{2m}{m}$$

Which is indeed true because $\binom{m}{2k+1} = 0$ as $k = m$.

Question 2.3. *Almost the same problem with equation (6), taking the coefficient of n^{2d+1} for an integer d in the range $\frac{m}{2} \leq d < m$, we get*

$$\mathbf{A}_{m,d} = 0$$

Let be $r = d$ *in* (4)

$$\mathbf{A}_{m,d} \frac{1}{(2d+1)\binom{2d}{d}} n^{2d+1} + 2\sum_{k} \mathbf{A}_{m,d} \frac{(-1)^d}{2d-2k} \binom{d}{2k+1} B_{2d-2k} n^{2k+1} = 0$$

Let be $d = m - 1$ *then again same principle*

$$2\sum_{k} \mathbf{A}_{m,d} \frac{(-1)^d}{2d-2k} \binom{d}{2k+1} B_{2d-2k} n^{2k+1} = 0$$

because $\binom{m-1}{2k+1} = 0$ *as* $k = m - 1$.

To summarize, the value of k should be in range $k \leq \frac{d-1}{2}$ so that binomial coefficient $\binom{d}{2k+1}$ is non-zero.

References

[1] Alekseyev, Max. MathOverflow answer 297916/113033, 2018. https://mathoverflow.net/a/297916/113033.

[2] Scheuer, Markus. MathStackExchange answer 4724343/463487, 2023. https://math.stackexchange.com/a/4724343/463487.

[3] Harry Bateman. *Higher transcendental functions [volumes i-iii]*, volume 1. McGRAW-HILL book company, 1953.

[4] Petro Kolosov. Entry A302971 in The On-Line Encyclopedia of Integer Sequences, 2018. https://oeis.org/A302971.

[5] Petro Kolosov. Entry A304042 in The On-Line Encyclopedia of Integer Sequences, 2018. https://oeis.org/A304042.

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