CENTRAL FACTORIAL NUMBERS - REFERENCES

PETRO KOLOSOV

Abstract. Mathematics Stack Exchange answer about the bibliography of Central Factorial Numbers, by Markus Scheuer. See

- https://math.stackexchange.com/a/3665722/463487

1. Introduction

Some references: We find in

- Combinatorial Identities by J. Riordan (1963) - [1, chapter 6.5 the formula (24)]

$$k!T(n,k) = \sum_{j=0}^{k} \binom{k}{j} (-1)^j \left(\frac{1}{2}k - j\right)^n \tag{24}$$

- The divided central differences of zero by L. Carlitz and J. Riordan (1961) - [2, formula (10a)]

$$K_{rs} = \frac{1}{(2s)!} \sum_{t=0}^{2s} (-1)^t \binom{2s}{t} (s-t)^{2r+2} \tag{10a}$$

- Interpolation by J. F. Steffensen (1927) - [3, Section 58]

The development of x^r in *central factorials*

$$x^r = \sum_{\nu=0}^{r} x^{[\nu]} \frac{\delta^\nu 0^r}{\nu!}$$

Date: December 23, 2025.

2010 *Mathematics Subject Classification.* 05A19, 05A10, 41A15, 11B83, 68W30.

Key words and phrases. Polynomial identities, Finite differences, Binomial coefficients, Faulhaber's formula, Sums of powers, Bernoulli numbers, Bernoulli polynomials, Interpolation, Discrete convolution, Combinatorics, Central factorial numbers, Central differences, OEIS.

leads to **central differences of nothing**, that is

$$\delta^m 0^r = \sum_{\nu=0}^{m} (-1)^\nu \binom{m}{\nu} \left(\frac{m}{2} - \nu\right)^r$$

Comment. The meaning of the left–hand side $\delta^m 0^r$ is given in the derivation below.

Here we show the derivation of (24) above following J. Riordan. It is based upon three ingredients: operators, a recurrence relation, and Newton's formula.

Operators. We recall the shift operator E^a and the difference operator Δ:

$$E^a f(x) = f(x+a),$$

$$\Delta f(x) = f(x+1) - f(x),$$

and introduce the *central difference operator* δ:

$$\delta f(x) = f\left(x + \frac{1}{2}\right) - f\left(x - \frac{1}{2}\right).$$

We can write the δ operator using shift and difference operators as

$$\delta f(x) = \left(E^{\frac{1}{2}} - E^{-\frac{1}{2}}\right) f(x) \tag{1}$$

$$= \Delta E^{\frac{1}{2}} f(x) = E^{\frac{1}{2}} \Delta f(x). \tag{2}$$

From (1), by successive application of δ, we obtain

$$\begin{aligned}\delta^k f(x) &= \left(E^{\frac{1}{2}} - E^{-\frac{1}{2}}\right)^k f(x) \\ &= \sum_{j=0}^{k} \binom{k}{j} (-1)^j E^{-\frac{j}{2}} E^{\frac{k-j}{2}} f(x) \\ &= \sum_{j=0}^{k} \binom{k}{j} (-1)^j f\left(x - j + \frac{k}{2}\right). \qquad (3)\end{aligned}$$

Note that (3) already has the shape of (24).

Central factorials. We denote by $x^{[n]}$ the *central factorial*, defined as

$$\begin{aligned} x^{[n]} &= x\left(x+\frac{n}{2}-1\right)^{\underline{n-1}} \\ &= x\left(x+\frac{n}{2}-1\right)\left(x+\frac{n}{2}-2\right)\cdots\left(x+\frac{n}{2}-n+1\right), \end{aligned}$$

where we use Knuth's notation for falling factorials $x^{\underline{n}} = x(x-1)\cdots(x-n+1)$.

The central factorials satisfy an important recurrence relation. Using (2) we obtain

$$\begin{aligned} \delta x^{[n]} &= \Delta E^{-\frac{1}{2}} x^{[n]} \\ &= \Delta \left(x-\frac{1}{2}\right)^{[n]} \\ &= \Delta \left(x-\frac{1}{2}\right)\left(x+\frac{n}{2}-\frac{3}{2}\right)^{\underline{n-1}} \\ &= \left(x+\frac{1}{2}\right)\left(x+\frac{n}{2}-\frac{1}{2}\right)^{\underline{n-1}} - \left(x-\frac{1}{2}\right)\left(x+\frac{n}{2}-\frac{3}{2}\right)^{\underline{n-1}} \\ &= \left(x+\frac{1}{2}\right)\left(x+\frac{n}{2}-\frac{1}{2}\right)\left(x+\frac{n}{2}-\frac{3}{2}\right)^{\underline{n-2}} \\ &\quad - \left(x-\frac{1}{2}\right)\left(x+\frac{n}{2}-\frac{3}{2}\right)^{\underline{n-2}}\left(x+\frac{n}{2}-\frac{3}{2}-n+2\right) \\ &= n x^{[n-1]}. \end{aligned} \tag{4}$$

This recurrence is analogous to $\frac{d}{dx}x^n = nx^{n-1}$.

Newton's formula. We expand $f(x)$ in central factorials and apply the operator δ:

$$f(x) = \sum_{n\geq 0} a_n x^{[n]},$$

$$\delta^j f(x) = \sum_{n\geq 0} a_n \delta^j x^{[n]} = \sum_{n\geq 0} a_n n^{\underline{j}} x^{[n-j]}, \tag{5}$$

$$\delta^j f(0) = \sum_{n\geq 0} a_n n^{\underline{j}} \delta_{n,j} = a_j j!. \tag{6}$$

From (6) we obtain Newton's formula in the form

$$f(x) = \sum_{j\geq 0} \frac{x^{[j]}}{j!}\, \delta^j f(0). \tag{7}$$

Finally, setting $f(x) = x^n$ in (7), denoting the coefficients by $T(n, k)$, and using (6), we obtain

$$x^n = \sum_{k=0}^{n} T(n,k) x^{[k]},$$
$$\delta^k 0^n = T(n,k) k!. \tag{8}$$

Using (3) in (8) gives

$$k! T(n,k) = \sum_{j=0}^{k} \binom{k}{j} (-1)^j \left(\frac{k}{2} - j\right)^n,$$

which is formula (24), and the claim follows.

References

[1] John Riordan. *Combinatorial identities*, volume 217. Wiley New York, 1968. https://www.amazon.com/-/de/Combinatorial-Identities-Probability-Mathematical-Statistics/dp/0471722758.

[2] L. Carlitz and John Riordan. The divided central differences of zero. *Canadian Journal of Mathematics*, 15:94–100, 1963. https://doi.org/10.4153/CJM-1963-010-8.

[3] Steffensen, Johan Frederik. *Interpolation*. Williams & Wilkins, 1927. https://www.amazon.com/-/de/Interpolation-Second-Dover-Books-Mathematics-ebook/dp/B00GHQVON8.

Version: `1.0.0+main.284fd00`

Sources: github.com/kolosovpetro/github-latex-template

Email: kolosovp94@gmail.com

Email address: kolosovp94@gmail.com

Software Developer, DevOps Engineer

URL: https://kolosovpetro.github.io