A NOVEL PROOF OF POWER RULE IN CALCULUS

PETRO KOLOSOV

Abstract. In Calculus, the power rule is a fundamental result stating that the derivative of a power function is given by the product of the exponent and the base raised to the power of the exponent minus one. Typically, the power rule is proven using the limit definition of the derivative alongside the Binomial theorem. In this manuscript, we present an alternative approach to proving the power rule, utilizing a specific polynomial identity that captures the function's growth. This method omits the direct use of the Binomial theorem, offering a distinct way to the same result.

Contents

1. Introduction and main results

Power rule can be considered as the one of the most fundamental rules in calculus. Most of us remember this law from the first calculus course

$$\frac{\mathrm{d}x^n}{\mathrm{d}x} = nx^{n-1}$$

Date: November 11, 2024.

2010 *Mathematics Subject Classification.* 26A24.

Key words and phrases. Calculus, Derivatives, Power rule, Binomial theorem, Differentiation.

Sources: https://github.com/kolosovpetro/ANovelProofOfPowerRuleInCalculus

where n is a constant. One of the common strategies to prove the power rule is by utilizing limit definition of derivative in conjunction with binomial theorem. Recall the limit form of derivative

$$\frac{\mathrm{d}f(x)}{\mathrm{d}x} = \lim_{h\to 0}\left[\frac{f(x+h)-f(x)}{h}\right]$$

where $f(x)$ is defined over $\mathbb{R}$ and at least of smoothness class C^1. Let be $f(x)=x^n$ with constant n. Then its derivative is

$$\frac{\mathrm{d}x^n}{\mathrm{d}x} = \lim_{h\to 0}\left[\frac{(x+h)^n-x^n}{h}\right]$$

Notice that we can express the function's growth $(x+h)^n$ by using binomial theorem

$$(x+h)^n = \sum_{k=0}^{n}\binom{n}{k}x^{n-k}h^k$$

So that

$$\begin{aligned}\frac{\mathrm{d}x^n}{\mathrm{d}x} &= \lim_{h\to 0}\left[\frac{1}{h}\sum_{k=1}^{n}\binom{n}{k}x^{n-k}h^k\right]\\ &= \lim_{h\to 0}\left[\binom{n}{1}x^{n-1}+\binom{n}{2}x^{n-2}h+\binom{n}{3}x^{n-3}h^2+\cdots+\binom{n}{n}x^0h^n\right]\\ &= \binom{n}{1}x^{n-1}\end{aligned}$$

However, is the binomial theorem the only polynomial identity to express the growth rate? Well, not really, we can utilize different approach to express polynomial growth. Consider the following identity for odd powers [1, 2, 3, 4]

$$(x-2a)^{2m+1} = \sum_{r=0}^{m}\mathbf{A}_{m,r}\sum_{k=a+1}^{x-a}(k-a)^r(x-k-a)^r \tag{1.1}$$

where $\mathbf{A}_{m,r}$ is a real coefficient defined recursively

$$\mathbf{A}_{m,r} = \begin{cases}(2r+1)\binom{2r}{r} & \text{if } r=m\\ (2r+1)\binom{2r}{r}\sum_{d\geq 2r+1}^{m}\mathbf{A}_{m,d}\binom{d}{2r+1}\frac{(-1)^{d-1}}{d-r}B_{2d-2r} & \text{if } 0\leq r<m\\ 0 & \text{if } r<0 \text{ or } r>m\end{cases} \tag{1.2}$$

where B_t are Bernoulli numbers [5] such that $B_1=\frac{1}{2}$. For example,

m/r	0	1	2	3	4	5	6	7
0	1							
1	1	6						
2	1	0	30					
3	1	-14	0	140				
4	1	-120	0	0	630			
5	1	-1386	660	0	0	2772		
6	1	-21840	18018	0	0	0	12012	
7	1	-450054	491400	-60060	0	0	0	51480

Table 1. Coefficients $\mathbf{A}_{m,r}$. See OEIS sequences [6, 7].

Properties of coefficients $\mathbf{A}_{m,r}$

- $\mathbf{A}_{m,m} = \binom{2m}{m}$
- $\mathbf{A}_{m,r} = 0$ for $m < 0$ and $r > m$
- $\mathbf{A}_{m,r} = 0$ for $r < 0$
- $\mathbf{A}_{m,r} = 0$ for $\frac{m}{2} \leq r < m$
- $\mathbf{A}_{m,0} = 1$ for $m \geq 0$
- $\mathbf{A}_{m,r}$ are integers for $m \leq 11$
- Row sums: $\sum_{r=0}^{m} \mathbf{A}_{m,r} = 2^{2m+1} - 1$

Therefore, by setting $a = -\frac{h}{2}$ in (1.1) we can express the growth rate of odd powers as

$$(x+h)^{2m+1} = \sum_{r=0}^{m} \mathbf{A}_{m,r} \sum_{k=-\frac{h}{2}+1}^{x+\frac{h}{2}} \left(k + \frac{h}{2}\right)^r \left(x - k + \frac{h}{2}\right)^r = \sum_{r=0}^{m} \mathbf{A}_{m,r} \sum_{k=1}^{x+h} k^r \left(x - k + h\right)^r$$

Thus,

$$\begin{aligned}\frac{\mathrm{d}x^{2m+1}}{\mathrm{d}x} &= \lim_{h\to 0}\frac{1}{h}\left[\sum_{r=0}^{m}\mathbf{A}_{m,r}\sum_{k=-\frac{h}{2}+1}^{x+\frac{h}{2}}\left(k+\frac{h}{2}\right)^r\left(x-k+\frac{h}{2}\right)^r-\sum_{r=0}^{m}\mathbf{A}_{m,r}\sum_{k=1}^{x}k^r(x-k)^r\right]\\ &= \lim_{h\to 0}\frac{1}{h}\left[\sum_{r=0}^{m}\mathbf{A}_{m,r}\left(\sum_{k=-\frac{h}{2}+1}^{x+\frac{h}{2}}\left(k+\frac{h}{2}\right)^r\left(x-k+\frac{h}{2}\right)^r-\sum_{k=1}^{x}k^r(x-k)^r\right)\right]\\ &= \lim_{h\to 0}\frac{1}{h}\left[\sum_{r=0}^{m}\mathbf{A}_{m,r}\left(\sum_{k=1}^{x+h}k^r\,(x-k+h)^r-\sum_{k=1}^{x}k^r\,(x-k)^r\right)\right]\end{aligned}$$

For instance, having $m = 1, 2$

$$\sum_{r=0}^{1}\mathbf{A}_{1,r}\sum_{k=-\frac{h}{2}+1}^{x+\frac{h}{2}}\left(k+\frac{h}{2}\right)^r\left(x-k+\frac{h}{2}\right)^r = h+x+(-1+h+x)(h+x)(1+h+x)$$

$$\sum_{r=0}^{2}\mathbf{A}_{2,r}\sum_{k=-\frac{h}{2}+1}^{x+\frac{h}{2}}\left(k+\frac{h}{2}\right)^r\left(x-k+\frac{h}{2}\right)^r$$

$$= h+x+(h+x)(1+h+x)(-1+h-h^2+h^3+x-2hx+3h^2x-x^2+3hx^2+x^3)$$

Which further simplifies to binomial theorem.

2. Conclusions

In this manuscript, we have presented an alternative approach to proving the power rule, utilizing a specific polynomial identity that captures the power function's growth. This method omits the direct use of the Binomial theorem, offering a distinct way to the same result. Results can be validated using Mathematica programs at GitHub repository github.com/kolosovpetro/ANovelProofOfPowerRuleInCalculus

References

[1] Kolosov, Petro. History and overview of the polynomial P(m,b,x), 2024. https://kolosovpetro.github.io/pdf/HistoryAndOverviewOfPolynomialP.pdf.

[2] Kolosov, Petro. On the link between binomial theorem and discrete convolution. *arXiv preprint arXiv:1603.02468*, 2016. https://arxiv.org/abs/1603.02468.

[3] Kolosov, Petro. 106.37 An unusual identity for odd-powers. *The Mathematical Gazette*, 106(567):509–513, 2022. https://doi.org/10.1017/mag.2022.129.

[4] Petro Kolosov. Polynomial identity involving Binomial Theorem and Faulhaber's formula, 2023. https://kolosovpetro.github.io/pdf/PolynomialIdentityInvolvingBTandFaulhaber.pdf.

[5] Harry Bateman. *Higher transcendental functions [volumes i-iii]*, volume 1. McGRAW-HILL book company, 1953.

[6] Petro Kolosov. Entry A302971 in The On-Line Encyclopedia of Integer Sequences. Published electronically at https://oeis.org/A302971, 2018.

[7] Petro Kolosov. Entry A304042 in The On-Line Encyclopedia of Integer Sequences. Published electronically at https://oeis.org/A304042, 2018.

Version: `1.0.1-tags-v1-0-0.10+tags/v1.0.0.bd5b0a6`

Software Developer, DevOps Engineer

Email address: kolosovp94@gmail.com

URL: https://kolosovpetro.github.io