A CURIOSITY ABOUT POLYNOMIAL INTERPOLATION

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ABSTRACT. Interpolation of cubes expected to be

$$n^3 = 6\binom{n}{3} + 6\binom{n}{2} + \binom{n}{1} + 0\binom{n}{0}$$

but got

$$n^3 = \sum_{k=1}^{n} \mathbf{A}_{m,0} k^0 (n-k)^0 + \mathbf{A}_{m,1} k^1 (n-k)^1$$

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1. Introduction

Interpolation is a process of finding new data points based on the range of a discrete set of known data points. Interpolation has been well-developed in between 1674–1684 by Issac Newton's fundamental works, nowadays known as foundation of classical interpolation theory [1].

The first time I found interpolation interesting was in 2016 when I observed a table of finite differences of cubes. Back then, I was a first-year mechanical engineering undergraduate. Due to my lack of mathematical knowledge, I started re-inventing interpolation formulas myself, fueled by pure passion and a sense of mystery. *All the mathematical laws and relations exist from the very beginning; we only reveal and describe them*, I thought. That mindset truly inspired me, and thus, my own mathematical journey began.

Consider finite differences of cubes n^3

n	n^3	$\Delta(n^3)$	$\Delta^2(n^3)$	$\Delta^3(n^3)$
0	0	1	6	6
1	1	7	12	6
2	8	19	18	6
3	27	37	24	6
4	64	61	30	6
5	125	91	36	
6	216	127		
7	343			

Table 1. Table of finite differences of the polynomial n^3.

The problem of interpolation of polynomials is a classical problem in mathematics and has been widely studied in literature. For instance, Concrete mathematics [2, p. 190] gives interpolation of cubes by using Newton's interpolation formula

$$n^3 = 6\binom{n}{3} + 6\binom{n}{2} + \binom{n}{1} + 0\binom{n}{0}$$

because

$$f(x) = \Delta^d f(0)\binom{x}{d} + \Delta^{d-1} f(0)\binom{x}{d-1} + \cdots + f(0)\binom{x}{0} = \sum_{r=0}^{d} \Delta^{d-r} f(0)\binom{x}{d-r}$$

However, interpolation of cubes can be also done in a different way. The key point that interpolation formula above iterates over the order d of finite difference. Alternatively, we can interpolate cubes n^3 as a sum of first order finite difference Δ as follows

$$n^3 = \Delta 0^3 + \Delta 1^3 + \Delta 2^3 + \cdots + \Delta(n-1)^3 = \sum_{k=0}^{n-1} \Delta k^3$$

We know that $\Delta^3 n^3 = 6$ is the constant for each n. The second difference of cubes $\Delta^2 n^3$ is a linear relation in terms of third order finite difference $\Delta^3 n^3$

$$\Delta^2 n^3 = (n+1)\Delta^3 n^3 = 6(n+1) \tag{1}$$

Finally, the first order finite difference Δn^3 is the following relation in terms of second order finite difference

$$\Delta n^3 = \Delta 0^3 + \Delta^2 0^3 + \Delta^2 1^3 + \cdots + \Delta^2 (n-1)^3 = 1 + \sum_{k=0}^{n-1} 6(k+1)$$

Altering summation bounds yields

$$\Delta n^3 = 1 + 6 \cdot 0 + 6 \cdot 1 + 6 \cdot 2 + 6 \cdot 3 + \cdots + 6 \cdot n = 1 + 6 \sum_{k=0}^{n} k \tag{2}$$

Therefore, we are able to express first order finite difference of cubes in form of sums as follows

$$\Delta(0^3) = 1 + 6 \cdot 0$$

$$\Delta(1^3) = 1 + 6 \cdot 0 + 6 \cdot 1$$

$$\Delta(2^3) = 1 + 6 \cdot 0 + 6 \cdot 1 + 6 \cdot 2$$

$$\Delta(3^3) = 1 + 6 \cdot 0 + 6 \cdot 1 + 6 \cdot 2 + 6 \cdot 3$$

Now it is time to assemble all the results above to get the polynomial n^3. Having the relation in cubes $n^3 = \Delta 0^3 + \Delta 1^3 + \Delta 2^3 + \cdots + \Delta(n-1)^3$ we get

$$n^3 = [1 + 6 \cdot 0] + [1 + 6 \cdot 0 + 6 \cdot 1] + [1 + 6 \cdot 0 + 6 \cdot 1 + 6 \cdot 2] \\ + \cdots + [1 + 6 \cdot 0 + 6 \cdot 1 + 6 \cdot 2 + \cdots + 6 \cdot (n-1)] \quad (3)$$

By rearranging the terms of the equation above, we get summation in terms of $k(n-k)$

$$n^3 = n + [(n-0) \cdot 6 \cdot 0] + [(n-1) \cdot 6 \cdot 1] + [(n-2) \cdot 6 \cdot 2] \\ + \cdots + [(n-k) \cdot 6 \cdot k] + \cdots + [1 \cdot 6 \cdot (n-1)] \quad (4)$$

By applying compact sigma sum notation yields an identity for cubes n^3

$$n^3 = n + \sum_{k=0}^{n-1} 6k(n-k) \quad (5)$$

The term n in the sum above can be moved under sigma notation, because there is exactly n iterations, therefore

$$n^3 = \sum_{k=0}^{n-1} 6k(n-k) + 1 \quad (6)$$

By inspecting the expression $6k(n-k) + 1$ we iterate under summation, we can notice that it is symmetric over k, let be $T(n,k) = 6k(n-k) + 1$, then

$$T(n,k) = T(n, n-k) \quad (7)$$

This symmetry allows us to alter summation bounds again, so that

$$n^3 = \sum_{k=1}^{n} 6k(n-k) + 1 \quad (8)$$

Curiously enough that although $\sum_{k=0}^{n-1} 6k(n-k) + 1$ and $\sum_{k=1}^{n} 6k(n-k) + 1$ both simplify to n^3, they produce different closed forms. Let be $P(n,q) = \sum_{k=0}^{q-1} 6k(n-k) + 1$ and

$Q(n,q) = \sum_{k=1}^{q} 6k(n-k) + 1$, then

$$P(n,q) = \begin{cases} q = 1: & 1 \\ q = 2: & -4 + 6n \\ q = 3: & -27 + 18n \end{cases}$$

$$Q(n,q) = \begin{cases} q = 1: & -5 + 6n \\ q = 2: & -28 + 18n \\ q = 3: & -81 + 36n \end{cases}$$

Now let's take a breath and briefly summarize all the results we got so far. So, we have successfully interpolated the polynomial n^3 using discrete set of finite differences data points applying the following algorithm

(1) Express second finite difference as linear relation in terms of third finite difference (1)
(2) Express first finite difference in terms of second (2)
(3) Express cubes as sum of first order finite differences in n (3)
(4) Rearrange the terms of the sum (4)
(5) Apply sigma notation (5)
(6) Move n under sigma (6)
(7) Apply symmetry (7)
(8) Alter summation bounds (8)

2. Generalizations

Assume that our previously obtained identities $n^3 = \sum_{k=0}^{n-1} 6k(n-k) + 1$ and $n^3 = \sum_{k=1}^{n} 6k(n-k) + 1$ have explicit form as follows

$$n^3 = \sum_{k} \mathbf{A}_{1,1} k^1 (n-k)^1 + \mathbf{A}_{1,0} k^0 (n-k)^0$$

where $\mathbf{A}_{1,1} = 6$ and $\mathbf{A}_{1,0} = 1$, respectively. Therefore, let be a conjecture

Conjecture 2.1. *For every $n \geq 1$, $n, m \in \mathbb{N}$ there are coefficients $\mathbf{A}_{m,0}, \mathbf{A}_{m,1}, \dots, \mathbf{A}_{m,m}$ such that*

$$n^{2m+1} = \sum_{k=1}^{n} \mathbf{A}_{m,0} k^0 (n-k)^0 + \mathbf{A}_{m,1}(n-k)^1 + \cdots + \mathbf{A}_{m,m} k^m (n-k)^m$$

Note that conjecture above assumes the convention $0^0 = 1$, reader may found a comprehensive discussion of it in [3].

Long story short, above conjecture is true, so that real coefficients $\mathbf{A}_{m,r}$ are following

m/r	0	1	2	3	4	5	6	7
0	1							
1	1	6						
2	1	0	30					
3	1	-14	0	140				
4	1	-120	0	0	630			
5	1	-1386	660	0	0	2772		
6	1	-21840	18018	0	0	0	12012	
7	1	-450054	491400	-60060	0	0	0	51480

Table 2. Coefficients $\mathbf{A}_{m,r}$. See OEIS sequences [4, 5].

These coefficients $\mathbf{A}_{m,r}$ are defined via a recurrence relation involving Binomial coefficients and Bernoulli numbers

Definition 2.2. *(Definition of coefficient $\mathbf{A}_{m,r}$.)*

$$\mathbf{A}_{m,r} = \begin{cases} (2r+1)\binom{2r}{r} & \text{if } r = m \\ (2r+1)\binom{2r}{r} \sum_{d \geq 2r+1}^{m} \mathbf{A}_{m,d} \binom{d}{2r+1} \frac{(-1)^{d-1}}{d-r} B_{2d-2r} & \text{if } 0 \leq r < m \\ 0 & \text{if } r < 0 \text{ or } r > m \end{cases}$$

where B_t are Bernoulli numbers [6]. It is assumed that $B_1 = \frac{1}{2}$. Properties of the coefficients $\mathbf{A}_{m,r}$

- $\mathbf{A}_{m,m} = (2m+1)\binom{2m}{m}$

- $\mathbf{A}_{m,r} = 0$ for $m < 0$ and $r > m$
- $\mathbf{A}_{m,r} = 0$ for $r < 0$
- $\mathbf{A}_{m,r} = 0$ for $\frac{m}{2} \leq r < m$
- $\mathbf{A}_{m,0} = 1$ for $m \geq 0$
- $\mathbf{A}_{m,r}$ are integers for $m \leq 11$
- Row sums: $\sum_{r=0}^{m} \mathbf{A}_{m,r} = 2^{2m+1} - 1$

Proof of conjecture (2.1) as well as other discussions on topics above can be found in literature [7, 8, 9, 10, 11]. Few OEIS sequences were contributed as well [12, 13, 14, 15, 16].

Very well, let's wrap up this technical section and move on to the more engaging discussions.

3. Discussions

3.1. **Interpolation.** Current manuscript starts from certain polynomial technique shown on base case of cubes, where the key identity is the tricky rearrangement of terms in the sum $n^3 = \sum_{k=0}^{n-1} \Delta k^3$ in (4). This rearrangement was done instead of applying Faulhaber's formula on $\Delta n^3 = 1+6\sum_{k=1}^{n} k$ which leads to well-known result involving Binomial theorem: $\Delta n^3 = 1 + 6\sum_{k=1}^{n} k = 1 + 6\frac{1}{2}(n + n^2) = 1 + 3n^3 + 3n$. In context of rearrangement (4) and moving n under summation (6)

Question 3.1. *Can we consider the process of finding identities for cubes* (6)*,* (8) *as an interpolation method?*

The steps (4) and (6) is the only distinct from well-known result

$$n^3 = \sum_{k=0}^{n-1} \sum_{r=0}^{2} \binom{3}{r} k^r$$

Instead, we arrived to identities

$$n^3 = \sum_{k=0}^{n-1} 6k(n-k) + 1; \qquad n^3 = \sum_{k=1}^{n} 6k(n-k) + 1$$

Question 3.2. *Assuming that question (3.1) is true, can we consider the conjecture (2.1) as an interpolation method for odd powers?*

3.2. **Literature.** The algorithm we used to obtain identities for cubes (6) and (8) is quite simple, if not naive. I believe it should be discussed in mathematical literature, as well as conjecture (2.1) that gives a set of real coefficients $\mathbf{A}_{m,r}$ such that

$$n^{2m+1} = \sum_{k=1}^{n} \mathbf{A}_{m,0} k^0 (n-k)^0 + \mathbf{A}_{m,1} (n-k)^1 + \cdots + \mathbf{A}_{m,m} k^m (n-k)^m$$

However, I was not able to find any references that in particular mention coefficients $\mathbf{A}_{m,r}$, which is one of open questions.

3.3. **Version for even powers.** We have the algorithm to obtain identities for cubes (6) and (8), as well as a set of real coefficients $\mathbf{A}_{m,r}$ such that

$$n^{2m+1} = \sum_{k=1}^{n} \mathbf{A}_{m,0} k^0 (n-k)^0 + \mathbf{A}_{m,1} (n-k)^1 + \cdots + \mathbf{A}_{m,m} k^m (n-k)^m$$

Is there analog for even powers?

3.4. **Relation with triangular numbers.** I have spotted that finite difference of cubes can be expressed in terms of triangular numbers, i.e.

$$\Delta n^3 = 1 + 6 \sum_{k=0}^{n} k = 1 + 6 \binom{n+1}{2}$$

where $\binom{n+1}{2}$ are triangular numbers, see OEIS sequence A000217 [17]. I wander is there a relation between Δn^5 and pentagonal numbers [18] as well...

Question 3.3. *Is there a relation between N-sided polygon numbers and finite differences of odd powers $2m+1$?*

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